SOLUTION OF TWO DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS USING FINITE DIFFERENCE METHOD

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ABSTRACT: *In this research paper, we find the analytical and numerical solutions of "Laplace's and Poisson's differential equations" satisfying the given boundary conditions, using numerical methods. Mainly, we focus on solution of two dimensional differential equations to understand the concept of Finite Difference Method (FDM).*

Keywords: *Laplace differential equation, Poisson' differential equation, Finite Difference Method.*

INTRODUCTION: In science and engineering fields, some problems are solved by using partial differential equations. A second order partial differential equation (PDE) is given by equation $P\frac{\partial^2 u}{\partial x^2} + Q\frac{\partial^2 u}{\partial x \partial y} + R\frac{\partial^2 u}{\partial y^2} + S\frac{\partial u}{\partial x} + T\frac{\partial u}{\partial y} + Uu + V = 0$ can be classified in different types.

The general equation of conic $Px^2 + Qxy + Ry^2 + Sx + Ty + U = 0$ is become the equations of Parabola, hyperbola and ellipse by the following criteria:

$$Q^2 - 4PR = \begin{Bmatrix} > 0, hyperbola \\ = 0, parabola \\ < 0, ellipse \end{Bmatrix}$$

For instance, we will manage an extraordinary sort elliptic equation termed Helmholtzs equation, that is expressed as

$$\frac{d^2\emptyset(x,y)}{dx^2} + \frac{d^2\emptyset(x,y)}{dy^2} + g(x,y)\emptyset(x,y) = f(x,y)$$

Over the domain $D = \{(x,y)/x0 \le x \le x_f , y0 \le y \le y_f \}$ with some boundary conditions.

$\emptyset(x, y_0) = b_{y_0}(x)$ $\qquad$ $\emptyset(x, y_f) = b_{y_f}(x)$

$\emptyset(x_0, y) = b_{x_0}(y)$ $\qquad$ $\emptyset(x_f, y) = b_{x_f}(y)$

The above equation is termed as Poisson's equation if $g(x,y) = 0$ and becomes Laplace's equation if $g(x,y) = 0$ and $f(x,y) = 0$

METHODOLOGY: In this section, we find the solution of 2-D PDE in engineering field (Aziz & Amin, 2016) and the extension of the result derived by (Goyal & Poonia 2021) The Poisson's equation representation in rectangular form given as,

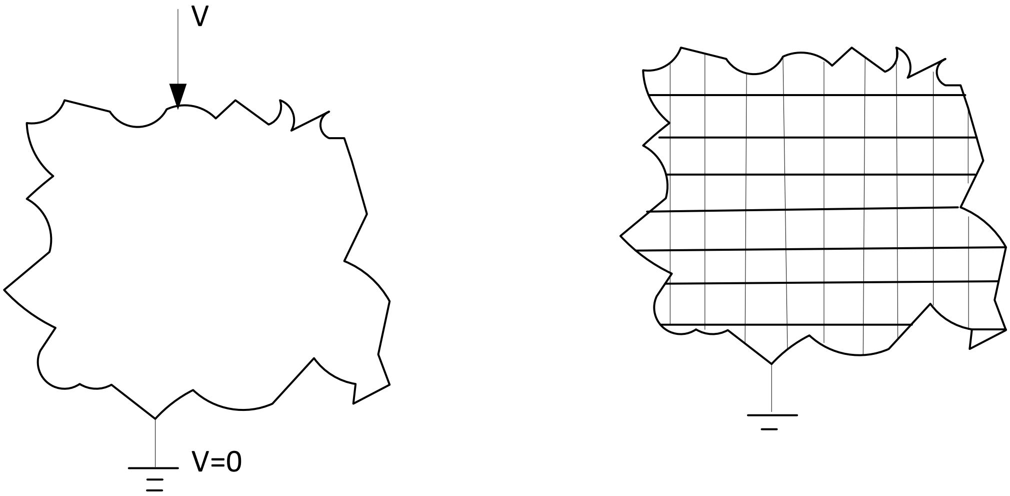
$$\nabla^2 V(x,y,z) = \frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = \frac{-\rho_v}{\varepsilon} \quad \ldots (1)$$

Assume, if electric potential (voltage) U, varies with respect to x and y coordinates only and is independent of z.coordinate. Thus, instead of a 3-D problem, now the problem converts into 2-D. Hence, equation (1) can be modified by replacing volume charge density $(\rho_v)$by surface charge density (ρ_s) as,

$$\nabla^2 V(x,y) = \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = \frac{-\rho_S}{\varepsilon} \quad \ldots (2)$$

where, ϵ denotes the permittivity of the medium.

The aim is to find the solution of $V(x,y)$ for region with boundary conditions as illustrated in Figure 1.1(a). First stage is to divide the solution area into finite number of grid as shown in Figure 1.1(b). The meshing can be either triangular, rectangular, or square, etc. In this problem, square and rectangular meshes are been discussed.



(a) (b)

Figure 1.1: Arbitrary region under study (b) Meshing of arbitrary region

Consider a grid(mesh) having dimensions a, b, c and d and electric potentials as,

$$V_1 = V(x, y+a), 2 = V(x-b, y), V_3 = V(x, y-c), V_4 = V(x+d, y)$$

and $V_0 = V(x, y)$ at the points (nodes) as depicted in Figure 1.1(c).

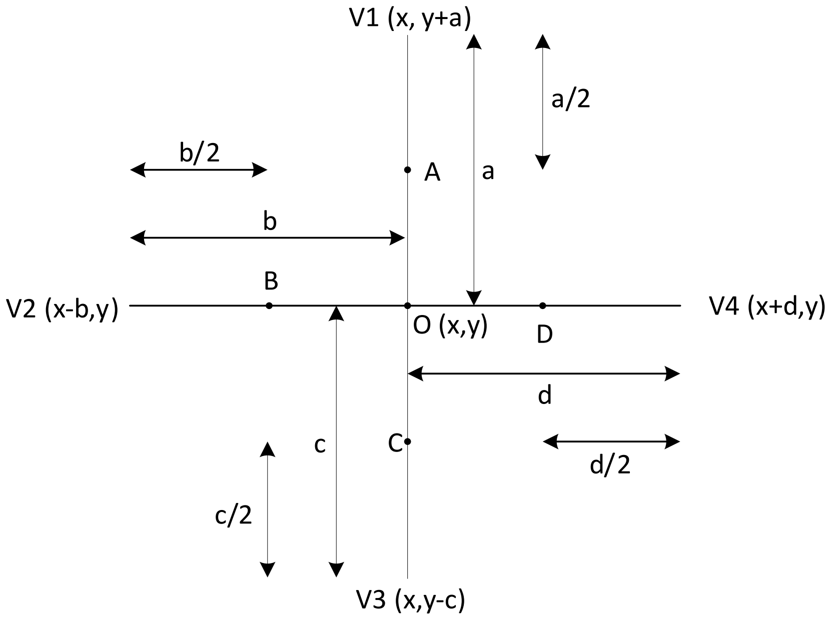


Figure 1.1 (c) Mesh configuration with unequal lengths

Now, by definition of first derivative of $V(x, y)$ at a points B and D, the derivative of V with respect to x can be given as,

$$(\frac{dV}{dx})_B = \frac{\Delta V}{\Delta x} = \frac{V_0 - V_2}{b} \qquad \ldots (3)$$

$$(\frac{dV}{dx})_D = \frac{\Delta V}{\Delta x} = \frac{V_4 - V_0}{d} \qquad \ldots (4)$$

Similarly, at nodes A and C the first derivatives are respectively expressed as,

$$(\frac{dV}{dx})_A = \frac{\Delta V}{\Delta y} = \frac{V_1 - V_0}{a} \qquad \ldots (5)$$

$$(\frac{dV}{dx})_C = \frac{\Delta V}{\Delta y} = \frac{V_0 - V_3}{c} \qquad \ldots (6)$$

Now the partial derivatives with order two of $V(x, y)$ at node 0 can be given as,

$$(\frac{d^2V}{dx^2})_0 = \frac{(\frac{dV}{dx})_D - (\frac{dV}{dx})_B}{\Delta x} = \frac{\frac{V_4 - V_0}{d} - \frac{V_0 - V_2}{b}}{\frac{d}{2} + \frac{b}{2}} \qquad \ldots (7)$$

We can write it as

$$(\frac{d^2V}{dx^2})_0 = \frac{2(V_4 - V_0)b - (V_0 - V_2)d}{bd\,(d+b)} \qquad \ldots (8)$$

$$\left(\frac{d^2V}{dy^2}\right)_0 = \frac{\left(\frac{dV}{dy}\right)_A - \left(\frac{dV}{dy}\right)_C}{\Delta y} = \frac{\frac{V_1-V_0}{a} - \frac{V_0-V_3}{c}}{\frac{a}{2}+\frac{c}{2}} \quad \ldots (9)$$

We can write the above equation as

$$\left(\frac{d^2V}{dy^2}\right)_0 = \frac{2(V_1-V_0)c-(V_0-3)a}{ac\,(a+c)} \quad \ldots (10)$$

Substituting the values of second order partial derivatives from equation (8) and equation (10) in equation (2) we have,

$$\frac{1}{a(a+c)}V_1 + \frac{1}{b(b+d)}V_2 + \frac{1}{c(a+c)}V_3 + \frac{1}{d(d+b)}V_4 - \left(\frac{1}{bd} + \frac{1}{ac}\right)V_0 = -\frac{\rho}{2\epsilon} \quad \ldots (11)$$

For the square grid/ mesh shape, equation (11) becomes,

$$\frac{1}{h^2}(V_1 + V_2 + V_3 + V_4 - 4V_0) = -\frac{\rho_S}{\varepsilon} \quad \ldots (12)$$

here, h is the grid or mesh size.

Equation (12) represents the approximation using finite difference for "Poisson's equation".

Now, if the solution area is charge free i.e. $\rho_s = 0$, then Poisson's equation becomes Laplace's equation Hence, equation (11) becomes,

$$\frac{1}{a(a+c)}V_1 + \frac{1}{b(b+d)}V_2 + \frac{1}{c(a+c)}V_3 + \frac{1}{d(d+b)}V_4 - \left(\frac{1}{bd} + \frac{1}{ac}\right)V_0 = 0 \quad \ldots (13)$$

and equation (23) becomes

$$\frac{1}{h^2}(V_1 + V_2 + V_3 + V_4 - 4V_0) = 0 \quad \ldots (14)$$

Rearranging equation (14) we have,

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4) \quad \ldots (15)$$

From Equation (15), it is clear that if the potential at the surrounding nodes is known then the V at unknown node can be calculated. Thus, the physical significance of the Laplace's equation is that the voltage at a node must be the average of the voltages at four surrounding nodes.

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