A THEOREM ON GENERALIZED$|v,\lambda|_k$ SUMMABILITY OF INFIINITE SERIES

Dr. Pragati Sinha
Associate Professor
Dept. of Applied Sc. &
Humanisites Mangalmay Institute of
Engg. & Tech.
Greater Noida, U.P. -201308
Email:
pragati.sinha.saxena@gmail.com
Dr. Sanjeev Kumar Saxena
Associate Professor
Department of Mathematics
N.M.S.N. Das (P.G.) College,
Budaun, U.P.-243601

ABSTRACT

1.1 DEFINITIONS AND NOTATIONS

Let Σa_n be a given infinite series$s_n$as its n^{th}partial sum. Let $\lambda = \{\lambda_n\}$ be a monotonic, non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$ and $\lambda_1 = 1$ The sequence-to-sequence transformation

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+1}}^{n} s_v$$

defines the generalized de la Vallee Poissin means of the sequence $\{s_n\}$ generated by the sequence $\{\lambda_n\}$. The series $\sum a_n$ is said to be summable $|V,\lambda|$ if the sequence $\{V_n(\lambda)\}$ is of bounded variation, that is to say

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda) - V_n(\lambda)| < \infty \qquad (1.1)$$

The series Σa_n will be said to be summable $|V,\lambda|_k, k \geq 1$, if the series

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} |V_{n+1}(\lambda) - V_n(\lambda)|^k < \infty.$$

For $\lambda_n = n$, it reduces to $|C, 1|_k$ and for $k = 1$ it is the same as $|v, \lambda|$.

If $\sum_{v=1}^{n} \frac{|s_V|}{V} = O(\log n)$ as $n \to \infty$

then Σa_n is said to be strongly bounded by logarithmic meanswith index 1 or simply bounded $[R, \log n, 1]$ we shall write throughout for any sequence $\{\epsilon_n\}$.

We shall have the occasions to write

$$\Delta\epsilon_n = \epsilon_n - \epsilon_{n+1}$$

$$\Delta^2\epsilon_n = \Delta(\Delta\epsilon_n)$$

A sequence $\{\epsilon_n\}$ is said to be convex if

$$\Delta^2\epsilon_n \geq 0, \quad n = 1,2,3, \ldots.$$

1.2 MISHRA and SHRIVASTAVA [2] have proved the following theorem:

Let $\{\psi_n\}$is a +ve nondecreasing sequence and $\{\beta_n\}$ and $\{\epsilon_n\}$ are such that

$$|\Delta\epsilon_n| \leq \beta_n \tag{1.2.1}$$

$$\beta_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{1.2.2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|\varphi_n < \infty \tag{1.2.3}$$

$$|\epsilon_n|\varphi_n = 0(1) \tag{1.2.4}$$

If

$$\sum_{v=1}^{n} \frac{|s_v|^k}{v} = O(\varphi_n), \text{ for } k \geq 1 \tag{1.2.5}$$

then $\Sigma a_n \epsilon_n |C, I|$ is summable.

Generalized the above theorem for $|V, \lambda|_k$ summability SHARMA and SINHA [3] have proved the following theorem:

THEOREM B: Let $\{\psi_n n\}$ be a +re nondecreasing sequence and there sequence $\{\beta_n\}$ and $\{\psi_n\}$ are such that

$$|\Delta\epsilon_n| \leq \beta_n \tag{1.2.6}$$

$$\beta_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{1.2.7}$$

$$\sum_{n=1}^{\infty} \lambda_n |\Delta\beta_n| \varphi_n < \infty \tag{1.2.8}$$

$$\text{If} \sum_{v=1}^{n} \frac{|s_v|^k}{\lambda_v} = 0(\psi_n) \tag{1.2.9}$$

for $k \geqslant 1$ then $\Sigma a_n \epsilon_n, |v, \lambda|_k$ is summable

where

$$\psi_n = \sum_{v=1}^{n} \lambda_v^{-1}.$$

The object of this paper is to generalized the theorem A and B.

1.3 THEOREM: Let $\{\psi_n\}$ is a +ve nondecreasing sequence and the sequence $\{\beta_n\}$ and $\{\in_n\}$ are such that

$$|\Delta\epsilon_{\mathrm{n}}| \leq \beta_{\mathrm{n}} \tag{1.3.0}$$

$$\beta_{\mathrm{n}} \longrightarrow 0 \text{ as } \mathrm{n} \longrightarrow \infty \tag{1.3.1}$$

$$\sum_{n=1}^{\infty} \lambda_n |\Delta\beta_n| \psi_n < \infty \tag{1.3.2}$$

$$\text{If} |\in_n| \psi_n = O(1) \tag{1.3.3}$$

$$\sum_{v=1}^{n} \frac{|s_v|^k}{v} = o(\psi_n \rho_n), k \geq 1 \tag{1.3.4}$$

where $\psi_n = \sum_{i=1}^{n} \lambda_1^{-1}$ and $\{\rho_n\}$ are such type of +ve non-decreasing sequence that

$$\lambda_n \psi_n \rho_n \Delta \left(\frac{1}{\rho_n}\right) = o(1) \quad \text{as} \quad n \to \infty \tag{1.3.5}$$

then$\sum \frac{a_n \epsilon_n}{\rho_n}, |v, \lambda|_k$is summable.

We shall use the following Lemma in the proof of our theorem.

1.4 Lemma [3], If $\{\psi_n\}, \{\beta_n\}$ and $\{\epsilon_n\}$ satisfies the conditions of the theorem then

$$\lambda_n \psi_n, \beta_n = O(1) \tag{1.4.1}$$

and

$$\sum_{n=1}^{\infty} \beta_n \psi_n < \infty \tag{1.4.2}$$

Proof of the Theorem

Let $$T_n = v_{n+1}(\lambda; \epsilon_n) - v_n(\lambda; \epsilon_n)$$

where$v_n(\lambda) \in$ is n^{th} De LaPoussian Pale mean.Then toprove the theorem we have

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} |I_n|^k < \infty.$$

Let $\Sigma' n$ is the summation which satisfying that $\lambda_{n+1} = \lambda_n$,and $\Sigma'' n$ runs over from $\lambda_{n+1} > \lambda_n$

then

$$T_n \frac{1}{\lambda_n \lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n+1} [(\lambda_{n+1} - \lambda_n) + \lambda_n] \frac{a_v \epsilon_v}{\rho_v}$$

when $\lambda_{n+1} = \lambda_n$ then we get

$$T_n = \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n+1} \frac{a_v \in_v}{\rho_v}$$

$$= \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n+1} \frac{va_v \in_v}{\rho_v}$$

Applying the Abel's transformation we have

$$T_n = [\Sigma_1 + \Sigma_2 + \Sigma_3]$$

where

$$\Sigma_1 = \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n} \Delta\left(\frac{\epsilon_v}{v\rho_v}\right) \sum_{r=0}^{v} ra_r$$

$$= \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n} \Delta\left(\frac{\epsilon_v}{v\rho_v}\right)(v+1)s_v$$

$$\Sigma_2 = \frac{1}{\lambda_{n+1}} \frac{\in_{n+1}}{(n+1)\rho_{n+1}} \sum_{r=0}^{n+1} ra_r$$

$$= \frac{1}{\lambda_{n+1}} \quad \frac{n+1}{\rho_{n+1}} s_{n+1}$$

and

$$\Sigma_3 = \frac{\epsilon_n - \lambda_{n+2}}{n+1(n-\lambda_{n+2})\rho_n - \lambda_{n+2}} \sum_{r=0}^{n-\lambda_{n+1}} ra_r$$

$$= \frac{n-\lambda_{n+2}s_n - \lambda_{n+2}}{n+1\lambda_n - \lambda_{n+2}}$$

Now we shall discuss at Σ_1 we have

$$\Sigma_1 = \Sigma_{11} + \Sigma_{12} + \Sigma_{13} + \Sigma_{14}$$

where

$$\Sigma_{11} = \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{\Delta\epsilon_v s_v}{\rho_v}$$

$$\Sigma_{12} = \frac{1}{\lambda n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{\Delta\epsilon_v s_v}{\rho_v}$$

$$\Sigma_{13} = \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{\Delta\epsilon_{v+1} s_v}{\rho_v}$$

and

$$\Sigma_{14} = \frac{1}{\lambda_{\mathrm{n}}} \sum_{v=n-\lambda_{n+2}}^{\mathrm{n}} \epsilon_{v+1} s_v \quad \Delta\left(\frac{1}{\rho_v}\right)$$

so we have to prove, by Bosanquet's inequality that

$$\Sigma^1 \lambda_n^{k-1} |\Sigma_{1r}|^k < \infty \text{ for } r = 1,2,3,4$$

$$\Sigma^1 \lambda_n^{k-1} |\Sigma_2|^k < \infty$$

$$\Sigma^1 \lambda_n^{k-1} |\Sigma_3|^k < \infty.$$

Now $\Sigma^1 \lambda_n^{k-1} |\Sigma_{11}|^k$

$$= \Sigma^1 \lambda_n^{k-1} \left| \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{\mathrm{n}} \Delta \frac{\epsilon_v s_v}{\rho_v} \right|^k$$

$$= O(1) \left[\Sigma^1 \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{\mathrm{n}} \frac{|\Delta\epsilon_v| |s_v|^k}{\rho_v} \right]$$

$$= O(1) \left[\Sigma^1 \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{\mathrm{n}} \frac{|\Delta\epsilon_v| \cdot |S_v|^k}{\rho_v} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\Delta\epsilon_{\mathrm{v}}|^{k-1}}{\rho_v} \right]$$

$$= O(1) \left[\Sigma^1 \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\Delta\epsilon_v| |s_v|^k}{f_v} \right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{|s_v|^k |\Delta\epsilon_v|}{\rho_v} \sum_{n=v}^{v+\lambda_v-1} \frac{1}{\lambda_n}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{|S_v|^k}{\rho_v}\beta_v\right]$$

Now

$$\sum_{v=1}^{m} \frac{|s_v|^k}{\lambda_v}\frac{\beta_v\lambda_v}{\rho_v} = \sum_{v=1}^{m-1} \Delta\left\{\frac{\beta_v\lambda_v}{v}\right\}\sum_{r=1}^{v} \frac{|s_r|^k}{r} + \frac{\lambda_m \quad {}^{\beta}m+1}{\rho_{m+1}}\sum_{v=1}^{m} \frac{|s_v|^k}{v}$$

$$= \Sigma_{11}^{(1)} + \Sigma_{11}^{(2)} + \Sigma_{11}^{(3)} + \Sigma_{11}^{(4)}$$

where

$$\Sigma_{11}^{(1)} = \sum_{v=1}^{m-1} \Delta\beta_v\lambda_v \quad \psi_v = O(1)$$

$$\Sigma_{11}^{(2)} = \sum_{\mathrm{v}=1}^{\mathrm{m}-1} \beta_{\mathrm{v}+1}\Delta\lambda_{\mathrm{v}} \quad \varphi_{\mathrm{v}} = O(1)$$

$$\overset{(3)}{\Sigma_{11}} = \sum_{v=1}^{m-1} \beta_{v+1}\lambda_{v+1}\Delta\left(\frac{1}{\rho_v}\right)\rho_v\psi_v = O(1)$$

and

$$\Sigma_{11}^{(4)} \Rightarrow \lambda_{\mathrm{m}}\beta_{\mathrm{m}+1}\varphi_{\mathrm{m}} = O(1) as m \to \infty$$

by the condition (1.3.1), (1.3.2), (1.3.3) and (1.3.4) and by the properties of Lemma. So,

$$\Sigma^1 \lambda_n^{k-1}|\Sigma_{11}|^k < \infty.$$

We discuss

$$\Sigma^1 \lambda_n^{k-1} \left| \sum_{12} \right|^k = \sum^{1} \lambda_n^{k-1} \left| \lambda_n \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\Delta\epsilon_v||s_v|}{v\rho_v} \right|^k$$

$$= O(1)\left[\Sigma^1 \frac{1}{\lambda_n}\left\{\sum_{v=n-\lambda_{n+2}}^{n} \frac{|\Delta\epsilon_v||s_v|^k}{\rho_v}\right\}\left\{\sum_{v=n-\lambda_n}^{n} \frac{\beta_v}{v\rho_v}\right\}^{k-1}\right]$$

$$= O(1)\left[\Sigma^1 \frac{1}{\lambda_n}\sum_{v=n-\lambda_{n+2}}^{n} \frac{\beta_v|s_v|^k}{v\rho_v}\right]$$

$$= O(1)\left[\Sigma^1 \frac{1}{\lambda_n}\sum_{v=n-\lambda_{n+2}}^{n} \frac{\beta_v|s_v|^k}{v\rho_v}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{\beta_v}{v}\frac{|s_v|^k}{\rho_v}\sum_{n=v}^{v+\lambda_v-1} \frac{1}{\lambda_n}\right]$$

Now

$$\sum_{v=1}^{m} \frac{\beta_v\lambda_v}{v\rho_v}\frac{|s_v|^k}{v} = \sum_{v=1}^{m-1} \Delta\{\frac{\beta_v\lambda_v}{v\rho_v}\}\sum_{r=1}^{v} \frac{|s_r|}{\lambda_r} + \frac{\beta_m\lambda_m}{m\rho_m}\sum_{r=1}^{m} \frac{|s_r|^k}{\lambda_r}$$

$$= O(1)\left[\Sigma_{12}^{(1)} + \Sigma_{12}^{(2)} + \Sigma_{12}^{(3)} + \Sigma_{12}^{(4)} + \Sigma_{12}^{(5)}\right]$$

where

$$\Sigma_{12}^{(1)} = \sum_{\mathrm{v}=1}^{\mathrm{m}-1} \frac{|\Delta\beta_{\mathrm{v}}|\lambda_{\mathrm{v}}\psi_{\mathrm{v}}}{\mathrm{v}} = \mathrm{O}(1)$$

$$\Sigma_{12}^{(2)} = \sum_{v=1}^{\mathrm{m}-1} \frac{\beta_{\mathrm{v}+1}\Delta\lambda_{\mathrm{v}}\varphi_{\mathrm{v}}}{\mathrm{v}} = O(1)$$

$$\Sigma_{12}^{(3)} = \sum_{v=1}^{m-1} \frac{\beta_{v+1}\lambda_{v+1}\psi_v}{v(v+1)} = O(1)$$

$$\Sigma_{12}^{(4)} = \sum_{\mathrm{v}=1}^{\mathrm{m}-1} \frac{\beta_{\mathrm{v}+1}\lambda_{\mathrm{v}+1}}{(\mathrm{v}+1)}\Delta\left(\frac{1}{\rho_{\mathrm{v}}}\right) \quad \rho_{\mathrm{v}}\psi_{\mathrm{v}} = 0(1)$$

and

$$\Sigma_{12}^{(5)} = \frac{\beta_m\lambda_m\psi_m}{m} = O(1)$$

as $m \longrightarrow \infty$ by the hypothesis of the theorem and the property of Lemma so,

$$\Sigma^1 \lambda_n^{k-1} |\Sigma_{12}|^k < \infty$$

Again,

$$\Sigma^1 \lambda_n^{k-1} |\Sigma_{13}|^k$$

$$= \Sigma^1 \lambda_n^{k-1} \left| \frac{1}{n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\epsilon_{v+1}||s_v|}{v\rho_v} \right|^k$$

$$= O(1) \left[\Sigma^1 \frac{1}{n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\epsilon_v||s_v|}{v\rho_v} \right]^k$$

$$= O(1) \left[\Sigma^1 \frac{1}{n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\in_v||s_v|^k}{v\rho_v} \left\{ \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\epsilon_v|}{v\rho_v} \right\}^{k-1} \right]$$

$$= O(1) \left[\Sigma^1 \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\epsilon_v||S_v|^k}{v\rho_v} \right]$$

$$= O(1) \left[\sum_{v=1}^{\infty} \frac{|\epsilon_v||s_v|^k}{v\rho_v} \sum_{v=n}^{v+\lambda_v-1} \frac{1}{\lambda_n} \right]$$

$$= O(1) \left[\sum_{v=1}^{\infty} \frac{|\in_v||S_v|^k}{v\rho_v} \right]$$

Now

$$\sum_{v=1}^{m} \frac{|\epsilon_v||s_v|^k}{v\rho_v} = \sum_{v=1}^{m} \frac{|\in_v|\lambda_v \mid}{v\rho_v} \frac{|s_v|^k}{\lambda_v}$$

$$= O(1) \left[\sum_{v=1}^{m-1} \frac{|\in_v||S_v|}{v\rho_v} \sum_{r=1}^{v} \frac{|S_r|^k}{r} + \frac{|\in_m|\lambda_m}{m\rho_m} \sum_{r=1}^{m} \frac{|S_r|^k}{r} \right]$$

where

$$= O(1)\left[\Sigma_{13}^{(1)} + \Sigma_{13}^{(2)} + \Sigma_{13}^{(3)} + \Sigma_{13}^{(4)} + \Sigma_{13}^{(5)}\right]$$

$$\Sigma_{13}^{(1)} = \sum_{v=1}^{m-1} \frac{\beta_v \lambda_v \psi_v}{v} = O(1)$$

$$\Sigma_{13}^{(2)} = \sum_{v=1}^{m-1} \frac{|\epsilon_{v+1}|\Delta\lambda_v \varphi_v}{v} = O(1)$$

$$\Sigma_{13}^{(3)} = \sum_{v=1}^{m-1} \frac{|\epsilon_{v+1}|\lambda_{v+1}\psi_v \rho_v\,|}{v} \quad \left(\frac{1}{\rho_v}\right) = O(1)$$

$$\Sigma_{13}^{(4)} = \sum_{v=1}^{m-1} \frac{|\epsilon_{v+1}|\lambda_{v+1}\psi_v}{v(v+1)} = O(1)$$

and

$$\Sigma_{13}^{(5)} = \frac{|\epsilon_m|\lambda_m \psi_m}{m} = O(1)$$

as$m \to \infty$ by the hypothesis and Lemmaso

$\varepsilon^1 \lambda_n^{k-1} |\Sigma_{13}|^k < \infty$

$\Sigma^1 \lambda_n^{k-1} |\Sigma_{14}|^k =$

$$= \Sigma^1 \lambda_n^{k-1} \left| \frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{n} \epsilon_{v+1}\ s_v \Delta\left(\frac{1}{\rho_v}\right) \right|^k$$

$$= O(1)\left[\Sigma^1 \frac{1}{n} \left\{ \sum_{v=n-\lambda_{n+2}}^{n} |\epsilon_v||s_v|\Delta\left(\frac{1}{\rho_v}\right) \right\}^k \right]$$

$$= O(1)\left[\Sigma^1 \frac{1}{\lambda_n} \left\{ \sum_{v=n-\lambda_{n+2}}^{n} |\epsilon_v||s_v|^k \Delta\left(\frac{1}{\rho_v}\right) \right\} \times\right.$$

$$\times\left\{\sum_{v=n-\lambda_{n+2}}^{n} |\epsilon_v| \quad \left(\frac{1}{\rho_v}\right)\right\}^{k-1}\Bigg]$$

$$= O(1) \quad \left[\Sigma^1 \frac{1}{n} \sum_{v=n-\lambda_{n+2}}^{n} \quad |\epsilon_v||s_v|^k \Delta\left(\frac{1}{\rho_v}\right)\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} |\epsilon_v||s_v|^k \Delta\left(\frac{1}{\rho_v}\right) \sum_{v=n-\lambda_{n+2}}^{v=\lambda_n-1} \frac{1}{\lambda_n}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} |\epsilon_v||s_v|^k \Delta\left(\frac{1}{\rho_v}\right)\right]$$

Now,

$$\sum_{v=1}^{m} |\epsilon_v||s_v|^k \Delta\left(\frac{1}{\rho_v}\right) = \sum_{v=1}^{m} |\epsilon_v|\lambda_v \Delta\left(\frac{1}{\rho_v}\right)\frac{|s_v|^k}{\lambda_v}$$

$$= O(1)\left[\sum_{v=1}^{m-1} \Delta\left\{|\epsilon_v|\lambda_v \Delta\left(\frac{1}{\rho_v}\right)\sum_{r=1}^{v} \frac{|S_r|^k}{\lambda_I} + \right.\right.$$

$$\left.\left. + |\epsilon_m|\lambda_m \Delta\left(\frac{1}{\rho_m}\right)\sum_{r=1}^{m} \frac{|s_r|^k}{\lambda_r}\right\}^3\right]$$

$$= O(1)\left[\Sigma_{14}^{(1)} + \Sigma_{14}^{(2)} + \Sigma_{14}^{(3)} + \Sigma_{14}^{(4)}\right]$$

where

$$\Sigma_{14}^{(1)} = \sum_{v=1}^{m-1} \beta_v \lambda_v \Delta\left(\frac{1}{\rho_v}\right) \quad \rho_v \psi_v = O(1)$$

$$\Sigma_{14}^{(2)} = \sum_{v=1}^{m-1} |\epsilon_{v+1}|\Delta\lambda_v \Delta\left(\frac{1}{\rho_v}\right)\psi_v = O(1)$$

$$\Sigma_{14}^{(3)} = \sum_{v=1}^{m-1} |\epsilon_{v+1}|\lambda_{v+1}\Delta^2\left(\frac{1}{\rho_v}\right)\rho_v \varphi_v = O(1)$$

$$\Sigma_{14}^{(4)} = |\epsilon_m|\lambda_m \Delta\left(\frac{1}{\rho_v}\right) \quad \rho_m \psi_m = O(1)$$

asm $\longrightarrow \infty$ by the hypothesis and Lemmaso,

$$\Sigma \lambda_n^{k-1} |\Sigma_{14}|^k < \infty$$

We get more than

$$\Sigma^1 \lambda_n^{k-1} |\Sigma_2|^k + \Sigma^1 \lambda_n^{k-1} |\Sigma_3|$$

$$= O(1)\left[\Sigma^1 \frac{|\epsilon_n||s_n|^k}{\lambda n |\rho_n|^k}\right]$$

$$= O(1)\left[\Sigma^1 \frac{|\epsilon_n||s_n|^n}{\lambda^n \rho n}\right]$$

Now,

$$\sum_1^m \frac{|\epsilon_n||s_n|^k}{\lambda_n \rho_n} = \sum_{n=1}^{m-1} \Delta\left(\frac{|\epsilon_n|}{\rho_n}\right) \sum_{r=1}^{n} \frac{|s_r|^k}{\lambda_r} + \frac{|\epsilon_m|}{m} \sum_{r=1}^{m} \frac{|s_r|^k}{\lambda_r}$$

$$= O(1)\left[\Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}\right]$$

where

$$\Sigma^{(1)} = \sum_{n=1}^{m-1} |\Delta \epsilon_n| \psi_n$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n \varphi_n$$

$$= O(1)$$

$$\Sigma^{(2)} = \sum_{n=1}^{m-1} |\epsilon_{n+1}| \Delta\left(\frac{1}{\rho_n}\right) \quad \rho_n \psi_n = o(1)$$

and

$$\Sigma^{(3)} = |\epsilon_{\mathrm{m}}|\varphi_{\mathrm{m}} = O(1)$$

as $\mathrm{m} \to \infty$ by the hypothesis of theorem and the property of Lemma. So,

$$\Sigma\lambda_{\mathrm{n}}^{\mathrm{k}-1}|\Sigma_2|^{\mathrm{k}} + \Sigma^1\lambda_{\mathrm{n}}^{\mathrm{k}-1}|\Sigma_3|^{\mathrm{k}} < \infty$$

by the reason

$$\Sigma^1\lambda_n^{k-1}|T_n|^k < \infty$$

where $\lambda_{n+1}>\lambda_n$, then we get

$$T_n = \frac{1}{\lambda_n\lambda_{n+1}}\sum_{v=n-\lambda_{n+1}}^{n+1}(\lambda_v + v - n - 1)\frac{\in_v va_v}{v\rho_v}$$

$$T_n = \frac{1}{\lambda_n\lambda_{n+1}}\sum_{v=n-\lambda_{n+1}}^{v}(\lambda_v + v - n - 1)va_v\frac{\in_v}{v\rho_v}$$

Applying the Abel's transformation we get

$$T_n = [\Sigma_1^1 + \Sigma_2^1 + \Sigma_3^1]$$

where

$$\Sigma_1^1 = \frac{1}{\lambda_{\mathrm{n}}2}\sum_{v=n-\lambda_{n+2}}^{n}\Delta\left\{(\lambda_n + v - n - 1)\frac{\epsilon_v}{v\rho_v}\right\}vs_v$$

$$\Sigma_2^1 = \frac{\epsilon_{n+1}s_{n+1}}{\lambda_{n+1}\rho_{n+1}}$$

and

$$\Sigma_3^1 = \frac{\epsilon_n - \lambda_{n+2}s_n - \lambda_{n+1}}{\lambda_n\lambda_n - \lambda_{n+1}\rho_{n-}\lambda_{n+2}}$$

So it is sufficient to show that

$$\Sigma''\lambda_n^{k-1}|\Sigma_r^l|^k < \infty \quad r = 1,2,3$$

we get

$$\sum{}'' \lambda_n^{k-1} |\Sigma_r'|$$

$$= \Sigma'' \frac{1}{\lambda_n^{k+1}} | \sum_{v=n-\lambda_{n+2}}^{v} \Delta \left\{ (\lambda_v + n - v - 1) \frac{\epsilon_v}{v\rho_v} \right\} v s_v |^k$$

$$\leq \sum{}'' \frac{1}{\lambda_n^{k+1}} \left[\sum_{v=n-\lambda_{n+2}}^{v} \left| \Delta \left\{ (\lambda_v + n - v - 1) \frac{\epsilon_v}{v\rho_v} \right\} v s_v \right|^k \right]$$

$$\left| \Delta \left\{ (\lambda_v + n - v - 1) \frac{\epsilon_v}{v\rho_v} \right\} \right| \leq \lambda_v \Delta \left(\frac{|\epsilon_v|}{v\rho_v} \right) + \left(\frac{|\epsilon_v|}{v\rho_v} \right)$$

such that

$$\Sigma \lambda_n^{k-1} \left| \Sigma_1^{1k} \right|^k = O(1) \quad [\Sigma_{11}' + \Sigma_{12}'']$$

where

$$\Sigma_{11}' = \Sigma'' \frac{1}{\lambda_n^{k+1}} \left\{ \sum_{r=n-\lambda_{n+2}}^{n} \lambda_n \, \Delta \left(\frac{|\epsilon_v|}{v\rho_v} \right) v |s_v|^k \right\}$$

$$\Sigma_{12}' = \Sigma'' \frac{1}{\lambda_n^{k+1}} \sum_{v=n-\lambda_{n+2}}^{\lambda_{n+2}} \frac{|\epsilon_v| |s_v|^k}{\rho_v}.$$

Now we discuss

$$\Sigma_{11}' = \Sigma_{11}'^{(1)} + \Sigma_{11}'^{(12)} + \Sigma_{11}'^{(3)}$$

where

$$\Sigma_{11}'^{(1)} = \Sigma'' \frac{1}{\lambda_n^{k+1}} \left\{ \sum_{v=n-\lambda_{n+2}}^{n} \frac{\lambda_v |\Delta\epsilon_v| |s_v|^k}{\rho_v} \right\}$$

$$= O\,(1)\, \Sigma'' \frac{1}{\lambda_n^{k+1}} \sum_{v=n-\lambda_{n+2}}^{n} \frac{\lambda_v \beta_v |s_v|^k}{\rho_v}$$

$$= O(1)\sum_{v=1}^{\infty}\frac{|s_v|^k\lambda_v \quad {}^{\beta}v}{\rho_v}\sum_{n\geq v}^{''}\frac{1}{\lambda_n^{k+1}}$$

$$= O(1)\sum_{v=1}^{\infty}\frac{|S_v|\beta_v}{\rho_v}$$

$= 0\ (1)$ as we have proved.

Now,

$$\Sigma_{11}^{'(2)} = \Sigma''\lambda_n^{\frac{1}{k+1}}\left\{\sum_{r=n-\lambda_{n+2}}^{n}\frac{|\epsilon_{v+1}||s_v|^k}{(v+1)\rho_v}\lambda_v\right\}$$

$$= O(1)\Sigma''\lambda_n^{\frac{1}{k+1}}\left\{\sum_{v=n-\lambda_{n+2}}^{n}\frac{|\epsilon_v||s_v|^k\lambda_v}{v\rho_v}\right\}$$

$$= O(1)\sum_{v=1}^{\infty}\frac{|\epsilon_v||s_v|^k}{v\rho_v}\lambda_v\sum_{n\geq v}^{''}\frac{1}{\lambda_n^{k+1}}$$

$$= O(1)\sum_{v=1}^{\infty}\frac{|\epsilon_v||S_v|^k}{v\rho_v} = O(1) \text{ as we have proved.}$$

Now,

$$\Sigma_{11}^{'(3)} = \Sigma''\frac{1}{\lambda_n^{k+1}}\sum_{v=n-\lambda_{n+2}}^{n}|\epsilon_{v+1}||s_v|^k\lambda_v\Delta\left(\frac{1}{\rho_v}\right)$$

$$= 0(1)\Sigma''\lambda_n^{\frac{1}{k+1}}\sum_{v=n-\lambda_{n+2}}^{n}\left\{|\epsilon_v||s_v|^k\lambda_v\Delta\left(\frac{1}{\rho_v}\right)\right\}$$

$$= 0(1)\sum_{v=1}^{\infty}|\epsilon_v||s_v|^k\lambda_v\Delta\left(\frac{1}{\rho_v}\right)\sum_{n\geq v}^{''}\lambda_n^{\frac{1}{k+1}}$$

$$= 0(1)\sum_{v=1}^{\infty}|\epsilon_v||s_v|^k\lambda_v\Delta\left(\frac{1}{\rho_v}\right) = O(1) \text{ as we have proved}$$

so,

$$\Sigma_{11}' = \Sigma_{11}'^{(1)} + \Sigma_{12}'^{(2)} + \Sigma_{13}'^{(3)} = 0(1)$$

Now

$$\Sigma_{12}' = \Sigma'' \frac{1}{\lambda_n^{k+1}} \sum_{v=n-\lambda_{n+2}}^{n} \frac{|\epsilon_v||s_v|^k}{\rho_v}$$

$$= O(1) \sum_{v=1}^{\infty} \frac{|\epsilon_v||s_v|^k}{\rho_v} \sum_{n\geq v}'' \frac{1}{\lambda_n^{k+1}}$$

$$= O(1) \sum_{v=1}^{\infty} \frac{|\epsilon_v||s_v|^k}{\rho_v}$$

$= O(1)$ as we have proved.

So,

$$\Sigma'' \quad {}_n^{k-1}|\Sigma_1'|^k = O(1)[O(1) + O(1)]$$

by this reason

$$\Sigma'' \lambda_n^{k+1}|\Sigma_1'|^k < \infty$$

Similarly,

$$\Sigma'' \lambda_n^{k+1}|\Sigma_2'|^k = O(1)$$

as we have proved already for Σ_2.

In the end

$$\Sigma'' \lambda_n^{k-1}|\Sigma_3'|^k = O(1)$$

as we have proved already for Σ_3. In this way proved theorem.

REFERENCES

[1] LEINDLER, L. : On the absolute summability factors of Fourier series, Acta.

Sci. Math., Szeged 28, 323-336 (1967).

[2] MISHRA, K.N. AND SHRIVASTAVA, R.S.L. : On absolute Cesaro summability factors of Infinite Series Portugaliae Math. Vol. 42 Fase 1 (1983-84).

[3] SHARMA, N.K, AND SINHA, R. : On $|v, \lambda|_k$ summability factors of Infinite series (in press).