BESSEL SEQUENCE AND FINITE NORMALIZED TIGHT FRAMES

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Abstract: *In this article, we show that a finite dimensional Hilbert space can have an infinite Bessel sequence, but a normalized Bessel sequence in a finite dimensional Hilbert space must be of finite length. A relation between the dimension of a given finite dimensional Hilbert space and the bound of any finite normalized tight frame for the underlying space is obtained. Also some properties of the frame operator and the Bessel sequence are discussed for finite normalized tight frame with some examples.*

Keywords: *Frame, Tight frame, Finite normalized tight frame, Bessel sequence, Frame operator*

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1 INTRODUCTION

Frames for Hilbert spaces were introduced by Duffin and Schaeffer (see [4]) in 1952 to study some deep problems in non-harmonic Fourier series. Duffin and Schaeffer abstracted the fundamental notion of Gabor [6] for signal processing. These ideas did not generate much interest outside of non-harmonic Fourier series and signal processing until the landmark paper [3] of Daubechies, Grossmannn, and Meyer in 1986. After this ground breaking work the theory of frames began to be widely studied. Frames are redundant sets of vectors in a Hilbert space, which yield one natural representation of each vector in the space, but may have infinitely many different representations for any given vector. It is this redundancy that makes frames useful in applications. Today, frames play an important role in many applications in mathematics, science, and engineering. Some of these applications include time-frequency analysis, internet coding, speech and music processing, communication, medicine, quantum computing, and many other areas.

In the recent past theory of frames has developed to broader areas. Tight frames are important in fast convergence and normalized tight frames control the elements of frames. The theory of normalized tight frames was developed by John Benedetto in [1].

In this paper, we discuss finite tight frames (FTF) and finite normalized tight frames (FNTF) and their relations with Bessel sequence. A Bessel sequence in finite dimensional Hilbert space is square-summable in norm and if a finite Bessel sequence which is a frame for a Hilbert space, then the dimension of the Hilbert space is finite. Our main result shows that a Bessel sequence in a finite dimensional Hilbert space must be of finite length. Further, we explicate some properties of Bessel sequence and frame operator with some examples.

2 DEFINITIONS AND PRELIMINARY RESULTS

We start with the definition of a frame.

Definition 2.1 *Let* $\mathcal{H}$ *be a finite dimensional Hilbert space over the field ,*$\mathbb{K}$ *where* $\mathrm{K} = \mathrm{R}$ *or* C*. A sequence* $\{x_n\}_{n=1}^{N}$ *called a frame for* H *if there exist finite positive constants* A *and* B*, such that*

$$(2.1) \qquad A\|x\|^2 \leq \sum_{k=1}^{N} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \ \textit{for all } x \in \mathcal{H}.$$

The positive constants A and B are called *upper* and *lower* frame bounds of $\{x_n\}_{n=1}^{N}$, respectively. They are not unique. The above inequality is called the *frame inequality* for the frame.

If the upper inequality in above inequality is satisfied, then we say that $\{x_n\}_{n=1}^N$ is a *Bessel sequence* for H with Bessel bound B.

Put

$$C_0 = \sup\{A > 0 : A \,\text{satisfies}\, (2.1)\}$$

and

$$D_0 = \sup\{B > 0 : B \,\text{satisfies}\, (2.1)\}$$

The numbers C_0 and D_0 are called *best bounds* or *optimal bounds* of the frame $\{x_n\}_{n=1}^N$.

Definition 2.2 *The operator* $\Theta : \mathbb{K}^N \to \mathbb{H}$ *given by*

$$\Theta(\{c_k\}) = \sum_{k=1}^{N} c_k x_k,$$

is called the pre-frame operator or the synthesis operator of the frame. The adjoint operator $\Theta^* : \mathbb{H} \to \mathbb{K}^N$ given by

$$\Theta^*(x) = \{\langle x, x_k \rangle\}_{k=1}^N,$$

and is called the analysis operator of the frame.

The frame operator of the frame is the operator $S = \Theta\Theta^* : \mathbb{H} \to \mathbb{H}$ which is given by

$$S(x) = \sum_{k=1}^{N} \langle x, x_k \rangle x_k, \text{ for all } x \in \mathbb{H}.$$

In term of the frame operator

$$\langle Sx, x \rangle = \sum_{k=1}^{N} |\langle x, x_k \rangle|^2, \; x \in \mathbb{H}.$$

Thus, the lower frame bound can thus be considered as some kind of 'lower bound' on the frame operator. The frame operator S is a positive, self-adjoint and invertible operator on H.

Definition 2.3 *A frame* $\{x_n\}_{n=1}^N$ *for* $\mathbb{H}$ *is said to be*

1. tight if $A = B$.
2. normalized tight or Parseval, if $A = B = 1$.

Definition 2.4 *An* $A-FNTF$ *for* $\mathbb{K}^d$ *is a finite sequence* $\{x_n : n = 1,\ldots,N\} \subseteq \mathbb{K}^d$ *for which the Euclidean norm* x_n *is* 1 *for each* x_n, *i.e.,* $\{x_n\}$ *is normalized, and for which*

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there exists $A > 0$ *such that* $\forall\, y \in \mathbb{K}^d$, $y = \frac{1}{A}\sum_{n=1}^{N}\langle y, x_n\rangle x_n$

Now we record some results which are useful for the better understanding of various tight frames. The following theorem was proved independently in [7] and [8].

Theorem 2.5 *Let* $\mathbb{H}$ *be a* d*-dimensional Hilbert space and* $N \geq d$. *Then, there exist a sequence of* N *elements which form an FNTF for* $\mathbb{H}$.

The following results not only give surety for the existence of FNTFs for $\mathbb{R}^2$ and $\mathbb{R}^3$ but also provide a direct computational method for finding some of them.

Theorem 2.6 *Normalized tight frames for* $\mathbb{R}^2$ *for* N *elements correspond to sequences* $\{z_n\}_{n=1}^{N} \subseteq \mathbb{C}$, *with* $|z_n| = 1$, *for all* n *and for which*

$$\sum_{n=1}^{N} z_n^2 = 0.$$

The authors of [5] and [7] proved the above result independently. It is observed that normalized tight frames for $\mathbb{R}^3$ which corresponds to a certain system satisfy additional conditions. This result is given in [5].

Theorem 2.7 *Normalized tight frames for* $\mathbb{R}^3$ *of* N *elements correspond to sequences* $\{z_n\}_{n=1}^{N} \subseteq \mathbb{C}$, *with* $|z_n| \leq 1$ *for all* n *that satisfy*

$$\sum_{n=1}^{N} |z_n|^2 = \frac{2}{3}N, \quad \sum_{n=1}^{N} z_n^2 = 0, \quad \sum_{n=1}^{N} z_n\sqrt{1-|z_n|^2} = 0.$$

3 MAIN RESULTS

Now we make an attempt of understanding of the normalized tight frames. In particular, we are interested in finite-dimensional Hilbert spaces $\mathbb{H}$, where $\mathbb{H} = \mathbb{C}^k$ or $\mathbb{H} = \mathbb{R}^k$, for a fixed positive integer k. Although it is quite possible for a finite dimension Hilbert space to have an infinite Bessel sequence, but a normalized Bessel sequence in a finite dimensional Hilbert space must be of finite length. In case of infinite sequence, the sum in equation (1) will be indexed over the set of natural numbers $\mathbb{N}$.

Proposition 3.1 *Let* $\mathbb{H}$ *be a Hilbert space and* $\{x_n\}_{n=1}^{\infty}$ *be a Bessel sequence in* $\mathbb{H}$. *If dimension of* $\mathbb{H}$ *is finite, then* $\{x_n\}_{n=1}^{\infty}$ *is square-summable in norm.*

Proof. Let $\mathbb{H}$ be a Hilbert space of dimension k (say) and $\{x_n\}_{n=1}^{\infty}$ be a Bessel sequence, then there exists a Bessel bound $B > 0$ such that

$$\sum_{k=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}.$$

Since orthonormal basis exists for every Hilbert space, let $\{e_i\}_{i=1}^{k}$ be the orthonormal basis of H. By Parseval's identity, we have

$$\|x_n\|^2 = \sum_{i=1}^{d} |\langle e_i, x_n\rangle|^2, \text{ for each } n = 1, 2, \ldots, N.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n\|^2 &= \sum_{n=1}^{\infty}\sum_{i=1}^{d} |\langle e_i, x_n\rangle|^2 \\ &= \sum_{i=1}^{d}\sum_{n=1}^{\infty} |\langle e_i, x_n\rangle|^2 \\ &\le \sum_{i=1}^{d} B\|e_i\|^2 \\ &= B.k\end{aligned}$$

Both B and k in the above expression are finite, so $\{\|x_n\|^2\}$ is square summable.

Proposition 3.2 *Let* H *be a Hilbert space and* $\{x_n\}_{n=1}^{\infty}$ *be a normalized Bessel sequence in* H*. If dimension of* H *is finite, then the sequence* $\{x_n\}_{n=1}^{\infty}$ *is a finite sequence.*

Proof. Let k be the dimension of the Hilbert space H and the Bessel sequence $\{x_n\}_{n=1}^{\infty}$ be normalized $\|x_n\| = 1$ i.e. , for every $n \in \mathrm{N}$. Then by Proposition 3.1, we have

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$$

For the convergence of the above series, $\|x_n\|^2$ must approaches to zero as n tends to infinity. But, $\|x_n\| = 1$ for all $n \in \mathrm{N}$. So the only possibility for $\{x_n\}_{n=1}^{\infty}$ is that it has only finite number of terms.

Proposition 3.3 *If* $\{x_n\}_{n=1}^{N}$ *is a finite Bessel sequence which is a frame for a Hilbert space* H*, then* $dim\mathrm{H} \le N$.

Proof. Let

$$\mathrm{M} = span\langle x_1, x_2, \ldots, x_N\rangle$$

be a subspace of the Hilbert space H. Let x be an element in $\mathrm{M}^{\perp}$, where $\mathrm{M}^{\perp}$ is the orthogonal complement of M in H, then $\langle x, x_n\rangle = 0$, for every $n = 1,2,\ldots,N$. This implies

$$\sum_{n=1}^{N} |\langle x, x_n \rangle|^2 = 0$$

Since the sequence $\{x_n\}_{n=1}^{N}$ is a frame, there exists a positive constant A such that

$$A\|x\|^2 \leq \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 = 0$$

As $A > 0$, so $\|x\|^2 = 0$ that is $x = 0$. Since x was arbitrary, $\mathcal{M}^{\perp} = 0$ and hence $\mathcal{M} = \mathcal{H}$. But $dim\mathcal{M} \leq N$, so $dim\mathcal{H} \leq N$.

Theorem 3.4 *Let S be a frame operator and $\{x_n\}_{n=1}^{\infty}$ be a Bessel sequence for a Hilbert space $\mathcal{H}$. Then, $\{x_n\}_{n=1}^{\infty}$ is an A-tight frame if and only if $S = AI$, where $I : \mathcal{H} \to \mathcal{H}$ is the identity mapping.*

Proof. The result follows directly by the definition of frame operator and A-tight frame. Let $S = AI$, then

$$S(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

$$\Leftrightarrow AI(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

$$\Leftrightarrow A.x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

$$\Leftrightarrow x = \frac{1}{A} \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

So, $\{x_n\}_{n=1}^{\infty}$ is an A-tight frame and conversely.

Theorem 3.5 *Let S be a frame operator and $\{x_n\}_{n=1}^{\infty}$ be a Bessel sequence for a Hilbert space $\mathcal{H}$. If $\{x_n\}_{n=1}^{\infty}$ is an A-NTF, then $A \geq 1$, and also $A = 1$ if and only if $\{x_n\}_{n=1}^{\infty}$ is orthonormal.*

Proof. Since $\{x_n\}_{n=1}^{\infty}$ is A-NTF i.e. $x = \frac{1}{A} \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$ and, $\|x_k\|$ for each $k = 1, 2, \ldots$, we have

$$
\begin{aligned}
A &= A\|x_k\|^2 \\
&= A\langle x_k, x_k\rangle \\
&= A\left\langle \frac{1}{A}\sum_{n=1}^{\infty}\langle x_k, x_n\rangle x_n, x_k \right\rangle \\
&= \sum_{n=1}^{\infty} |\langle x_k, x_n\rangle|^2 \\
&= \|x_k\|^2 + \sum_{n\neq k}^{\infty} |\langle x_k, x_n\rangle|^2 \\
&= 1 + \sum_{n\neq k}^{\infty} |\langle x_k, x_n\rangle|^2
\end{aligned}
$$

Thus, $A \geq 1$. In the above expression, if $A=1$, then $\sum_{n\neq k}^{\infty} |\langle x_k, x_n\rangle|^2 = 0$, this implies $\langle x_k, x_n\rangle = 0$, for $x_k \neq x_n$. This means $\{x_n\}_{n=1}^{\infty}$ is an orthogonal sequence. As, $\|x_k\| = 1$so $\{x_n\}_{n=1}^{\infty}$ is orthonormal. Conversely if $\{x_n\}_{n=1}^{\infty}$ is orthonormal, then $\sum_{n\neq k}^{\infty} |\langle x_k, x_n\rangle|^2 = 0$ and hence $A=1$.

The following result gives a unique choice for the bound of FNTF for underlying space.

Theorem 3.6 *If* H *d-dimensional Hilbert space and* $\{x_n\}_{n=1}^{N}$ *is an* A-$FNTF$*, then* A *is the ratio of number of elements in the frame to the dimension of the dimension of the Hilbert space* H.

Proof. Since every Hilbert space has an orthonormal basis, so let us assume $\{e_j\}_{j=1}^{d}$ be an orthonormal basis for H. Then

$$Ad = A\sum_{j=1}^{a} \|e_j\|^2$$

By definition of $A-FNTF$ frame, $e_j = \frac{1}{A}\sum_{n=1}^{N}\langle e_j, x_n\rangle x_n$, foreach $j = 1,2,\ldots,d$. So

$$Ad = \sum_{j=1}^{d}\sum_{n=1}^{N} |\langle e_j, x_n\rangle|^2 = \sum_{n=1}^{N}\sum_{j=1}^{d} |\langle e_j, x_n\rangle|^2$$

By Parseval's identity, we have $\|x_n\|^2 = \sum_{j=1}^{-} |\langle e_j, x_n\rangle|^2$, so

$$Ad = \sum_{n=1}^{N} \|x_n\|^2 = N.$$

Therefore, $A = \frac{N}{d}$, which completes the proof.

From the above theorem is easy to see that if $\{x_n\}_{n=1}^{N}$ is an A-$FNTF$ then $A = 1$ if and only if $\{x_n\}_{n=1}^{N}$ is an orthonormal basis of the Hilbert space H.

4 EXAMPLES AND MOTIVATION

We discuss now some examples of FNTF in finite dimensional Hilbert spaces. First we take $\mathsf{H} = \mathsf{R}^2$. By Theorem 2.5, there exist an FNTF for H of N elements where $N \geq 2$.

First we consider the case $N > 2$. The N vectors in R^2 identified with N^{th} roots of unity are an FNTF. Let $\{x_n\}_{n=1}^{N}$ are the N^{th} roots of unity, i.e., $\{x_n\}_{n=1}^{N}$ satisfies the following equation

$$z^N = 1.$$

If N is odd, then the set obtained by squaring the elements of the set of the N^{th} roots of unity is the same set. Indeed, if we choose $N = 3$, then $1, \omega$, ω^2 are cube root of unity, where $\omega^3 = 1$.

Since

$$1^2 = 1, \omega^2 = \omega^2 \text{and} (\omega^2)^2 = \omega^4 = \omega^3\omega = \omega.$$

Therefore, we obtained the same set $\{1, \omega, \omega^2\}$.

Now we consider the case when N is even. One may observed that by taking the square of the elements of the set containing all the N^{th} roots of unity, we obtain two copies of the $\frac{N}{2}$-th roots of unity. This can be understood by taking $N = 4$. Consider the set $\{1, -1, i, -i\}$.

Since

$$1^2 = 1, -1^2 = 1, i^2 = 1, -i^2 = 1.$$

Therefore, $\{1, -1\}$ and $\{1, -1\}$ are two copies of the $4/2 = 2^{th}$ roots of unity. We also observe that in both the situations, the sum of the squares of the N^{th} roots of unity is 0.

Now discuss the case when $N = 2$. The N^{th} roots of unity are 1 and -1 do not form an FNTF for H. Let, if possible $\{1, -1\}$ form an FNTF for $\mathsf{H} = \mathsf{R}^2$. Then, by Theorem 2.6, we have

$$(1)^2 + (-1)^2 \neq 0.$$

This is a contradiction.

Remark 4.1 *The family* $\{1, i\}$ *is not only FNTF but also form ONB for* $\mathbb{R}^2$ *as*

$$1^2 + i^2 = 0.$$

Now we discuss a system of finite vectors in $H = \mathbb{R}^3$ to be an FNTF. Any finite normalized tight frame for $\mathbb{R}^2$ whose elements also sum to zero may be converted into a FNTF for $\mathbb{R}^3$. Indeed, let $\{z_n\}_{n=1}^N \subseteq S^1$ satisfies

$$\sum z_n = \sum z_n^2 = 0.$$

Then, we can show that $\{\sqrt{\frac{2}{3}} z_n\}_{n=1}^N$ is FNTF for $\mathbb{R}^3$.

We compute

$$\sum_{n=1}^N |\sqrt{\frac{2}{3}} z_n|^2 = \sum_{n=1}^N \frac{2}{3} | z_n |^2$$

$$= \frac{2}{3} \sum_{n=1}^N | z_n |^2$$

$$= \frac{2}{3} N. \qquad (4.1)$$

and

$$\sum_{n=1}^N (\sqrt{\frac{2}{3}} z_n)^2 = \sum_{n=1}^N \frac{2}{3} (z_n)^2$$

$$= \frac{2}{3} \sum_{n=1}^N (z_n)^2$$

$$= 0. \qquad (4.2)$$

By using as $| z_n | = 1$, we obtain

$$\sum_{n=1}^N (\sqrt{\frac{2}{3}} z_n) z_n \sqrt{1 - \frac{2}{3} | z_n |^2} = \sum_{n=1}^N \sqrt{\frac{2}{3}} z_n \sqrt{1 - 2/3}$$

$$= \sum_{n=1}^N \sqrt{\frac{2}{3}} z_n \sqrt{\frac{1}{3}}$$

$$= \frac{\sqrt{2}}{3} \sum_{n=1}^N z_n$$

$$= 0. \qquad (4.3)$$

By using equations (4.1), (4.2) and (4.3) in Theorem 2.7, we conclude that $\{\sqrt{\frac{2}{3}}z_n\}$ is FNTF for H.

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