ON THE CONVERGENCE OF AN IMPLICIT LINEAR MULTISTEP METHOD OF ORDER SIX FOR THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: *This paper presents the convergence of an implicit linear multistep method of order six for the solution of ordinary differential equations. The derivation of this method is based on the interpolation and collocation methods. Numerical Examples were taken into consideration to determine the accuracy of the method. The method is convergent and stable.*

Keywords: *Convergence, Collocation Method, Implicit Linear Multistep Method, Interpolation Method, Ordinary Differential Equations*

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1.0 INTRODUCTION

It has been discovered that mathematical models resulting into single or system of first order ordinary differential equations are largely applied in nearly all discipline most especially in Sciences, Engineering and Economics. Any system whose behavior can be modeled by first order ordinary differential equations can be solved numerically to any desired degree of accuracy. Numerical solution of ordinary differential equations remain an active field of investigation, though, the area of research vary significantly.

Linear multistep method is a computational procedure whereby a numerical approximation x_{n+1} to the exact solution $x(t_{n+1})$ of the first order initial value problem of the form

$$x' = g\left(t, x\right), x(t_0) = x_0 \tag{1}$$

The general linear multistep method is given by

$$\sum_{k=0}^{j} \alpha_k x_{n+k} = h \sum_{k=0}^{j} \beta_k g_{n+k} \tag{2}$$

Where α_k and β_k are constants, h is the step size. It is assumed that the function $g\left(t, x\right)$ is Lipschitz continuous throughout the interval $a \le t \le b$. Equation (2) includes Simpson method, Adam Bashforth and Adam Molton Methods. All Adam's methods are regarded as constant coefficient method but in this paper, linear multistep method with constant coefficient of higher step number j is generated. The parameters of this method are determined by the collocation approach in which the approximate solution is determined from the condition that the equation must be stratified at certain given point. It involves the determination of an approximate solution in a suitable set of function called the basis function.

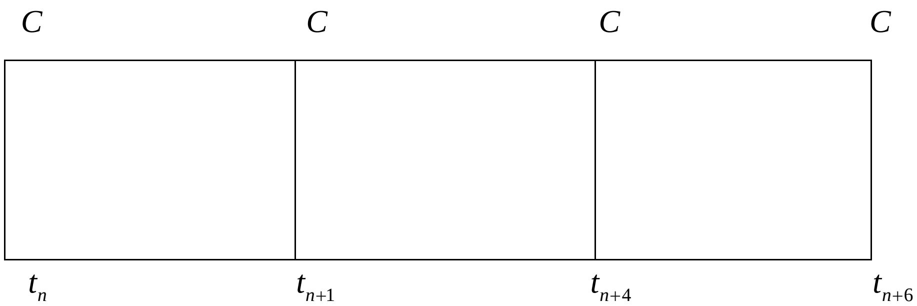
A number of researchers have developed linear multistep methods for the solution first order initial value problems in ordinary differential equations. These include [1], [2], [3], [4], [5], [6] just to mention few.

In this work we shall consider the derivation of an implicit linear multistep method of order six and some numerical examples to discuss the convergence of the method.

2.0 DERIVATION OF AN ORDER SIX LINEAR MULTISTEP METHOD

Now we shall derive an order six implicit linear multistep method for the solution of first order differential equation using collocation and interpolation methods.

Collocation points are used to collocate the different system. The interpolation points are used to interpolate the approximate solution with the diagram below. Both Collocation and interpolation are done at all even points $t = t_n, t_{n+2}$ and t_{n+4} while evaluation is done at $t = t_{n+6}$.



Next, we shall present the derivation of the scheme as follows,

2.1 Derivation of the Implicit Scheme

The basis function is given by:

$$x(t) = \sum_{k=1}^{6} b_k t_k \tag{3}$$

Equation (3) is needed in the derivation of the scheme for solving first order differential equation.

Expanding (3) we have,

$$x(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 \tag{4}$$

Differentiating (4) with respect to t

$$x'(t) = b_1 + 2b_2 t + 3b_3 t^2 + 4b_4 t^3 + 5b_5 t^4 + 6a_6 t^5 \tag{5}$$

Collocating (5), we have that at $t = t_n, t_{n+2}, t_{n+4}$ and t_{n+6}

Therefore,

$$g_n = b_1 + 2b_2 t_n + 3b_3 t_n^2 + 4b_4 t_n^3 + 5b_5 t_n^4 + 6b_6 t_n^5 \tag{6}$$

$$g_{n+2} = b_1 + 2b_2 t_{n+2} + 3b_3 t_{n+2}^2 + 4b_4 t_{n+2}^3 + 5b_5 t_{n+2}^4 + 6b_6 t_{n+2}^5 \tag{7}$$

$$g_{n+4} = b_1 + 2b_2 t_{n+4} + 3b_3 t_{n+4}^2 + 4b_4 t_{n+4}^3 + 5b_5 t_{n+4}^4 + 6b_6 t_{n+4}^5 \tag{8}$$

$$g_{n+6} = b_1 + 2b_2 t_{n+6} + 3b_3 t_{n+6}^2 + 4b_4 t_{n+6}^3 + 5b_5 t_{n+6}^4 + 6b_6 t_{n+6}^5 \tag{9}$$

Interpolating at the points $t = t_n, t_{n+2}$ and t_{n+4}, then

$$x_n = b_0 + b_1 t_n + b_2 t_n^2 + b_3 t_n^3 + b_4 t_n^4 + b_5 t_n^5 + b_6 t_n^6 \tag{10}$$

$$x_{n+2} = b_0 + b_1 t_{n+2} + b_2 t_{n+2}^2 + b_3 t_{n+2}^3 + b_4 t_{n+2}^4 + b_5 t_{n+2}^5 + b_6 t_{n+2}^6 \quad (11)$$

$$x_{n+4} = b_0 + b_1 t_{n+4} + b_2 t_{n+4}^2 + b_3 t_{n+4}^3 + b_4 t_{n+4}^4 + b_5 t_{n+4}^5 + b_6 t_{n+4}^6 \quad (12)$$

Using Gaussian Elimination method to determine the values of the coefficients $b_{0,} b_1, b_2, b_3, b_4, b_5$ and b_6, then we have that:

$$b_0 = x_n - b_1 t_n - b_2 t_n^2 - b_3 t_n^3 - b_4 t_n^4 - b_5 t_n^5 - b_6 t_n^6 \quad (13)$$

$$b_1 = g_n - 2b_2 t_n - 3b_3 t_n^2 - 4b_4 t_n^3 - 5b_5 t_n^4 - 6b_6 t_n^5 \quad (14)$$

$$\begin{aligned} b_2 &= \frac{1}{4h^2}(x_{n+2} - x_n) - \frac{1}{2h} g_n - b_3(3t_n + 2h) - b_4(6t_n^2 + 8t_n h + 4h^2) \\ &- b_5(10t_n^3 + 20t_n^2 + 2o_n th + 8h^3) - b_6(15t_n^4 + 40t_n^3 h + 60t_n^2 h + 48t_n h^3 + 16h^4) \end{aligned} \quad (15)$$

$$b_3 = \frac{1}{4h^2}(g_{n+2} + g_n) - \frac{1}{4h^3}(x_{n+2} - x_n) - b_4(4t_n + 4h) - b_5(10t_n^2 + 20t_n h + 12h^2) \quad (16)$$

$$b_4 = \frac{1}{64h^4}(x_{n+4} + 4g_{n+2} - 5x_n) - \frac{1}{16h^3}(2g_{n+2} + g_n) - b_5(5t_n + 8h) - b_6(15t_n^2 + 48t_n h + 44h^2) \quad (17)$$

$$b_5 = \frac{1}{64h^4}(x_{n+4} + 4g_{n+2} + g_n) - \frac{3}{128h^5}(x_{n+4} - x_n) - b_6(6t_n + 12h) \quad (18)$$

$$b_6 = \frac{1}{4224h^5}(g_{n+6} - 24g_{n+4} - 57g_{n+2} - 10g_n) + \frac{1}{2816h^5}(19g_{n+4} - 8g_{n+2} - 11g_n) \quad (19)$$

By inserting the coefficient b_0 into (1), evaluating at $t = t_{n+6}$, substituting equations (14), (15), (16), (17), (18) and (19) for b_1, b_2, b_3, b_4, b_5 and b_6 respectively and simplify we have the scheme:

$$x_{n+6} + \frac{27}{11} x_{n+4} - \frac{27}{11} x_{n+2} - x_n = \frac{6}{11} h(g_{n+6} + 9g_{n+4} + g_n) \quad (20)$$

Equation (20) is called implicit linear multistep method of order six.

3.0 NUMERICAL EXAMPLES

Here we present some numerical examples to discuss the convergence of the method.

Example 1

Consider the first order initial value problem of the form

$$x' = 2tx, x(0) = 1, t \in (0,1), h = \frac{1}{100} \quad (21)$$

Whose exact solution is given by

$$x(t) = e^{t^2} \quad (22)$$

The result obtained is shown in Table 1 below.

Table 1: Result for the Example 1, Using the Method (20)

N	t_n	x_n	$x(t_n)$	$e_n = \lvert x(t_n) - x_n \rvert$
1	0.1	1.0101	1.0100	0.0001
2	0.2	1.0408	1.0408	0.0000
3	0.3	1.0942	1.0943	0.0001
4	0.4	1.1735	1.1734	0.0001
5	0.5	1.2839	1.2840	0.0000
6	0.6	1.4333	1.4332	0.0001
7	0.7	1.6321	1.6323	0.0002
8	0.8	1.8964	1.8965	0.0001
9	0.9	2.2480	2.2479	0.0001
10	1.0	2.7183	2.7184	0.0001

Example 2

Next, we consider the first order initial value problem of the form

$$x' = 1 + x^2, x(0) = 1, t \in (0,1), h = \frac{5}{100} \tag{23}$$

This has an exact solution given below

$$x(t) = \tan\left(t + \frac{\pi}{4}\right) \tag{24}$$

The result obtained is displayed in Table 2 below.

Table 2: Result for the Example 2, Using the Method (20)

N	t_n	x_n	$x(t_n)$	$e_n = \lvert x(t_n) - x_n \rvert$
1	0.1	1.2230	1.2230	0.0000
2	0.2	1.5084	1.5085	0.0001
3	0.3	1.8958	1.8958	0.0000
4	0.4	2.4649	2.4650	0.0001
5	0.5	3.4082	3.4082	0.0000
6	0.6	5.3319	5.3318	0.0001
7	0.7	11.6814	11.6814	0.0000
8	0.8	-68.4797	-68.4798	0.0001
9	0.9	-8.6876	-8.6876	0.0000
10	1.0	-4.5880	-4.5879	0.0001

4.0 DISCUSSION OF RESULTS

The error is obtained, using different method which is the difference between the exact solution and the approximate solution. The above results are obtained using Q-BASIC programming language.

As we can see from Tables 1 and 2, with step sizes $h = \frac{1}{100}$ and $\frac{5}{100}$ respectively, the error values are very small for the method. Hence the method is stable, consistent, convergent and better in accuracy.

5.0 CONCLUSION

In this paper, an implicit linear multistep of order six for the solution of first order initial value problems in ordinary differential equations is derived. The derivation is based on collocation-interpolation method.

It is observed that, as the step size reduces the more accurate is the method. This method provides the closest accurate value for the solution of any differential equation of first order.

REFERENCES

[1] Awoyemi, D. O. (1992): On Some Continuous Linear Multistep Methods, for Initial Value Problems, Ph.D. Thesis (Unpublished), University of Ilorin, Nigeria.

[2] Collatz, L. (1960): Numerical Treatment of Differential Equations, Springer Verlag Berlin.

[3] Gear, C. W. (1991): Numerical Initial Value Problems in Ordinary Differential Equation, Prentice Hall Englewood Cliffs, New Jersey.

[4] Henrici, P. (1962): Discrete Variable Methods in Ordinary Differential Equation, John Wiley and Sons, New York.

[5] Kayode, S. J. (2004): A Class of Linear Multistep Collocation Methods for Discrete Solution of Second Order Differential Equation, Ph.D. Thesis (Unpublished), Federal University of Technology, Akure, Nigeria.

[6] Lambert, J. D. (1976): Computational Methods in Ordinary Differential Equation, John Wiley and Sons, New York.