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DIRECT AND INDIRECT ADJOINT APPROACH FOR AN OPTIMAL CONTROL PROBLEM WITH FIRST AND HIGHER ORDER COMPLEMENTARY SLACKNESS – INEQUALITY CONSTRAINTS

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Abstract: *In this paper, the direct and indirect adjoining approach with complementary slackness first order constraints and higher order constraints with continuous adjoint functions solving optimal control problem by adjoint approach. We also support this approach by suitable relevant example*

Key Words: *Adjoint Approach, Switching time and switching functions, Growth conditions, Pure and mixed state inequality constraints*

1. Introduction

The main focus of this article how do solve optimal problem by direct and indirect adjoint approach?

Optimal control (OC) deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. Any control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that maximize (minimize) the cost functional. The previous many articles deals with OC derived by Pontryagain's Maximum Principle (PMP) or by Hamilton Jacobean equations Method (HJM). This paper deals with adjoint approach. OC is the heart of many optimization applications in different areas, particularly in engineering and economics [7], [12]

1.2 Some preliminary concepts, assumptions, definitions and theorems

Optimal Control Problem (OCP) is described by a number of parameters, consider

$x = (x_1, x_2, \dots, x_n)$, which evolves according to a state equation,

$\dot{x}(t) = g\big(t, x(t), u(t)\big)\ where\ u = (u_1, u_2, \dots, u_n)$, represents the control exercised on the system. This control vector should satisfies various types of constraints depending on the nature of the problem, in this paper we only consider the restriction $u(t) \in U_{adj} \subseteq \mathbb{R}^m\ \forall\, t$, the state equation is also complemented with initial or final condition such as $x(0) = x_0\ and\ x(t) = x_T, where\ T$, it is the time horizon what we are considering, and the objective functional measuring how good a given control $u\ is$

The form of objective function is $\max F(x,u) = \int_0^T f\left(t, x(t), u(t)\right)\, where\, f: [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$, measures the rate how good a given control $u\, is$? A pair (x,u), it is said to be **feasible or admissible**, if the following conditions are satisfied [9]

a. Constraints on the control $u(t) \in U_{adj} \subseteq \mathbb{R}^m\ \forall\, t \in [0,T]$
b. State Law $\dot{x}(t) = g\big(t, x(t), u(t)\big) \forall\, t \in [0,T]$
c. End point conditions $x(0) = x_0\ and\ x(t) = x_T$, and then the OCP is

$$\max F(x,u) = \int_0^T f\left(t, x(t), u(t)\right)$$

Subject to $\quad u(t) \in U_{adj}$

$$\dot{x}(t) = g\big(t, x(t), u(t)\big)$$

With $\quad x(0) = x_0\ and\ x(t) = x_T$

1.3 The Hamiltonian and Multipliers

Let $\max F(x,u) = \int_0^T f\left(t, x(t), u(t)\right)$, for all pairs (x,u), such that $\dot{x}(t) = g\big(t, x(t), u(t)\big)$, together with appropriate conditions at the end points, but the state equation may be considered as a point wise constraint that can be treated by introducing a multiplier or co state $\lambda(t)$

Consider the co state function $\lambda: [0,T] \longrightarrow \mathbb{R}^n$, and equation above are given the Lagrangian problem of the following

$$L(x,u,\lambda,\dot{x}) = \int_0^T \left[f\big(t, x(t), u(t)\big) + \lambda(t)\left(g\big(t, x(t), u(t)\big)\right) - \dot{x}(t)\right] dt \longrightarrow (1)$$

$$\text{Take } G(x,u,\lambda,\dot{x}) = \left[f\big(t, x(t), u(t)\big) + \lambda(t)\left(g\big(t, x(t), u(t)\big)\right) - \dot{x}(t)\right] \longrightarrow (2)$$

From (2) we get Euler Lagrangian equations system can be derived by

$$\frac{\partial}{\partial x} G(x,u,\lambda,\dot{x}) = \frac{\partial}{\partial x}\left[f\big(t, x(t), u(t)\big) + \lambda(t)\left(g\big(t, x(t), u(t)\big)\right) - \dot{x}(t)\right]$$

$$\Longrightarrow \frac{\partial}{\partial x} G(x,u,\lambda,\dot{x}) = f_x\left[\big(t, x(t), u(t)\big) + \lambda(t)\left(g_x\big(t, x(t), u(t)\big)\right)\right] = 0 \longrightarrow (3)$$

$$\frac{\partial}{\partial u} G(x,u,\lambda,\dot{x}) = \frac{\partial}{\partial u}\left[f\big(t, x(t), u(t)\big) + \lambda(t)\left(g\big(t, x(t), u(t)\big)\right) - \dot{x}(t)\right]$$

$$\Longrightarrow \frac{\partial}{\partial u} G(x,u,\lambda,\dot{x}) = f_u\left[\big(t, x(t), u(t)\big) + \lambda(t)\left(g_u\big(t, x(t), u(t)\big)\right)\right] = 0 \longrightarrow (4)$$

$$\frac{\partial}{\partial \lambda} G(x,u,\lambda,\dot{x}) = \frac{\partial}{\partial \lambda}\left[f\big(t, x(t), u(t)\big) + \lambda(t)\left(g\big(t, x(t), u(t)\big)\right) - \dot{x}(t)\right]$$

$$\Longrightarrow \frac{\partial}{\partial \lambda} G(x,u,\lambda,\dot{x}) = g\big(t, x(t), u(t)\big) - \dot{x}(t) = 0 \longrightarrow (5)$$

$$\frac{\partial}{\partial \dot{x}} G(x,u,\lambda,\dot{x}) = \frac{\partial}{\partial \dot{x}}\left[f\big(t, x(t), u(t)\big) + \lambda(t)\left(g\big(t, x(t), u(t)\big)\right) - \dot{x}(t)\right]$$

$$\Longrightarrow \frac{\partial}{\partial \dot{x}} G(x,u,\lambda,\dot{x}) = -\lambda(t) = 0 \longrightarrow (6)$$

From (3) to (6) we get the following three equations

$$\frac{d}{dt}[-\lambda(t)] = f_x\left[f\big(t, x(t), u(t)\big) + \lambda(t)\Big(g_x\big(t, x(t), u(t)\big)\Big)\right]$$

$$\Longrightarrow f_x\left[f\big(t, x(t), u(t)\big) + \lambda(t)\Big(g_x\big(t, x(t), u(t)\big)\Big)\right] + \dot{\lambda}(t) = 0 \longrightarrow (7)$$

$$f_u\left[\big(t, x(t), u(t)\big) + \lambda(t)\Big(g_u\big(t, x(t), u(t)\big)\Big)\right] = 0 \longrightarrow (8)$$

$$g\big(t, x(t), u(t)\big) - \dot{x}(t) = 0 \longrightarrow (9)$$

Equations (7) to (9) determined the conditions for the control to maximize (minimize) the objective functions

Definition – 1.4:

The control Hamiltonian function $\mathcal{H}$, of the OCP is defined as:

$$\mathcal{H}: [0,\infty) \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n^*} \longrightarrow \mathbb{R}\ with\ \mathcal{H}(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

Using this definition, and we rewrite the three equations (7), (8) and (9)

a. Adjoint Condition
   $\dot{\lambda}(t) = -\frac{\partial}{\partial x}\mathcal{H}[t, x(t), u(t), \lambda(t)]$
b. Optimality Condition
   $\frac{\partial}{\partial u}\mathcal{H}[t, x(t), u(t), \lambda(t)] = 0$
c. State equation
   $\dot{x}(t) = g\big(t, x(t), u(t)\big)$
d. Transversals conditions
   If we set $x(T)$, to be free then we have the following conditions corresponding to the bounder (Initial value) case problem $\lambda(T) = 0$

These four conditions are necessary conditions for an OCP [9]

Theorem – 1.5

Let $f\ and\ g$, they are linear convex functions in $(x, u), \forall\ fixed\ t \in [0, T]$. Then every solution of the system of optimality with the appropriate end point conditions including transversality will be an optimal solution of the control problem [9]

1.6. Pontryagain's maximum or minimum principle (PMP)

PMP is used in OCP to find the best possible control for taking a system from one state to another, particularly in the presence of constraints for the state or input control. This principle sates informally that the Hamiltonian must be minimized or maximized over U, the set of all permissible controls.

If $u^* \in U$, it is the optimal control for the problem, then the principle states that

$\mathcal{H}[t, x^*(t), u^*(t), \lambda^*(t)] \leq \mathcal{H}[t, x(t), u(t), \lambda(t)], \forall\, u \in U, t \in [0, T]\ where\ x^* \in C^{(1)}[0, T]$, it is the optimal state trajectory and $\lambda^* \in [0, T]$, it is the optimal co state trajectory and considers the OCP in maximum form:

$$\max F\,(x, u) = \int_0^T f\,\big(t, x(t), u(t)\big)dt + \mathrm{S}(x(T), T)$$

Subject to $\quad \dot{x}(t) = g\big(t, x(t), u(t)\big), x(0) = x_0, where\ \mathrm{S}(x(T), T)$, and it is known as the **salvage functions**.

Suppose that $x(t), u(t)$, represent the state trajectory and optimal control respectively and then there exists an adjoint $\lambda(t)$satisfies the following conditions [9]

a. Adjoint Condition
   $\dot{\lambda}^*(t) = -\frac{\partial}{\partial x}\mathcal{H}[t, x^*(t), u^*(t), \lambda^*(t)]$
b. State equation
   $\dot{x}^*(t) = g\big(t, x^*(t), u^*(t)\big)\ with\ \ x(0) = x_0$
c. Transversals conditions
   $\lambda^*(t) = S_x(x^*(T), T)$
d. The maximum condition
   $\mathcal{H}[t, x^*(t), u^*(t), \lambda^*(t)] \leq \mathcal{H}[t, x(t), u(t), \lambda(t)]$

1.7. Bounded Control

Let the OCP [8]

$$\max F\,(x, u) = \int_0^T f\,\big(t, x(t), u(t)\big)dt + \mathrm{S}(x(T), T)$$

Subject to $\quad \dot{x}(t) = g\big(t, x(t), u(t)\big), x(0) = x_0$

With $\quad a \leq u(t) \leq b$, and hence the conditions for optimality for bounded controls are given:

a. State equation
   $\dot{x}(t) = g\big(t, x(t), u(t)\big)\ with\ x(0) = x_0$
b. Adjoint Condition
   $\dot{\lambda}(t) = -\frac{\partial}{\partial x}\mathcal{H}[t, x(t), u(t), \lambda(t)]$
c. Transversals conditions
   $\lambda(t) = S_x(x(T), T)$
d. The optimality Condition

$$\begin{cases} u^* = a\ \ if\ \frac{\partial}{\partial u}\mathcal{H}[t, x(t), u(t), \lambda(t)] < 0 \\ a < u^* < b\ if\ \frac{\partial}{\partial u}\mathcal{H}[t, x(t), u(t), \lambda(t)] = 0 \\ u^* = b\ \ if\ \frac{\partial}{\partial u}\mathcal{H}[t, x(t), u(t), \lambda(t)] > 0 \end{cases}$$

Definition – 1.8:

The point $t_i's$, at which the control switches between the minimum and the maximum is called the **switching time** and if the Hamiltonian problem is [8]

$\mathcal{H}[t, x(t), u(t), \lambda(t)] = f_1(t,x) + uf_2(t,x) + \lambda(t)[g_1(t,x) + ug_2(t,x)]$, that is

$\mathcal{H}[t, x(t), u(t), \lambda(t)] = f_1(t,x) + \lambda(t)g_1(t,x) + u(t)[f_2(t,x) + \lambda(t)g_2(t,x)]$, and then

a. It contains no information about u, and then the function $\psi(t) = \frac{\partial}{\partial x}\mathcal{H}[t, x(t), u(t), \lambda(t)]$, it can be zero at some finite number of $t_i^{'}s$ and also the optimality Condition
$\begin{cases} u^* = a \ \ if \ \psi(t) < 0 \\ a < u^* < b \ if \ \psi(t) = 0 \\ u^* = b \ \ if \ \psi(t) > 0 \end{cases}$, and hence we get
$\psi(t) = f_2(t,x) + \ \lambda(t)g_2(t,x)$, it is called the **switching function**

Definition – 1.9:

A control $u(t) \in U_{adj}$, it is called $\boldsymbol{Bang\ Bang}$, if for each $t \in [0,T], and\ each\ index\ i = 1, \dots, m$, we have $|u_i(t)| = 1 \ where\ u(t) = \big(u_1(t), \dots, u_m(t)\big)$ [8]

2. Analysis of constrained OCP

OCP with state variable inequality constraints are an important role in mechanics, aerospace, management science and economics. These problems are not solved easily and even the theory is not unambiguous, since, there are various forms of the necessary and sufficient optimality condition. More specially, we deal with problems with both pure and mixed state variable constraints. Pure constrains are inequality constraints expressed only in terms of the state variables and possibly time. Mixed constraints are constraints on control variables that may depend on the state variables and time [10]

1.1 Problems with mixed inequality constraints

OCP with state inequality constraint arise frequently in practical applications. Consider the problem to find a piecewise continuous control $u^* \in C[0,T]$ with associated response $x^* \in C^{(1)}[0,T]$, and a terminal time $T^* \in [0,T]$, such that the following the constraints are satisfied and the cost function takes on its maximum value

$\max F = \int_0^T f\,(t,x,u)dt$

Subject to $\dot{x}(t) = g\big(t, x(t), u(t)\big), x(0) = x_0, x(T) = x_T$

With $h\big(t, x(t), u(t)\big) \leq 0$

Assume that the components of $h\big(t, x(t), u(t)\big)$ depend explicitly on the control u and the following constraint qualification condition holds

$$\left(\frac{\partial}{\partial u}h, diag\ (h)\right) \longrightarrow \qquad (10)$$

It is full rank. In other words, the gradient with respect to u , of all the active constraint $h\big(t, x(t), u(t)\big)$, it must be linearly independent. Possible ways of attempting to solve OCP with mixed inequality constraints are to form a Lagrangian function L, by adjoining $h\big(t, x(t), u(t)\big)$ to the Hamiltonian function $\mathcal{H}$, with Lagrangian multiplier vector function μ [3]

That is $L(t,x,u,\lambda,\ \mu) = \mathcal{H}(t,x,u,\lambda) + \mu h \mathcal{H}(t,x,u)$ *where*

$\mathcal{H}(t,x,u,\lambda) = f\big(t,x(t),u(t)\big) + \lambda g\big(t,x(t),u(t)\big)$

2.2.1. **Necessary conditions for optimality**

$\max F(x,u) = \int_0^T f\big(t,x(t),u(t)\big)dt$

Subject to $\dot{x}(t) = g\big(t,x(t),u(t)\big), x(0) = x_0, x(T) = x_T$

With $h\big(t,x(t),u(t)\big) \le 0$

And also with fixed time and free terminal time and where f, g *and* h they are continuously differentiable with respect to (t,x,u) *on* $[0,\infty) \times \mathbb{R}^m \times \mathbb{R}^n$ and suppose that $u^* \in C[0,T]$, it is a maximum for the problem and let x^* denotes the optimal response. If the constraints qualification conditions are hold for every $t \in [0,T]$, and then we have

a. The function $\mathcal{H}[t,x^*(t),u^*(t),\lambda^*(t)]$ attains its maximum on $U(x^*(t),t)$ *at* $u = u^*(t), \forall\, t \in [0,T]$ *and also* $\mathcal{H}[t,x^*(t),u^*(t),\lambda^*(t)] \ge \mathcal{H}[t,x^*(t),u(t),\lambda(t)], \forall\, u \in U[(x^*(t),t)]$ *where* $U[(x(t),t)] = \big\{u(t) \in \mathbb{R}^n \,|\, h\big(t,x(t),u(t)\big) \le 0\big\}$
b. The quadruple (t,x^*,u^*,λ^*) satisfies the equations $\dot{x}^*(t) = L(t,x,u,\lambda,\mu);\ \dot{\lambda}^*(t) = L_x(t,x,u,\lambda,\mu)$ *and* $L_u(t,x,u,\lambda,\mu) = 0$, at each instant t of continuity of u^*
c. The vector function μ^*, it is continuous at each instant of continuity of u^* and satisfies $\mu(t)h\big(t,x(t),u(t)\big) \le 0$ *for* $\mu(t) \ge 0$

2.2.2 **Extension to General State of Terminal Constraints**

The maximum principle given in above conditions can be extended to the case where general terminal constraints are specified on the state variables as

$a(x(T),T) \ge 0$ *and* $b(x(T),T) = 0$, and a terminal term is added to the cost functional as

$\max F(x,u) = \int_0^T f\big(t,x(t),u(t)\big)dt + S(x(T),T),$ *where* a, b *and* S , they are continuously differentiable with respect to (t,x) *for all* $(t,x) \in [0,T] \times \mathbb{R}^m$, and suppose that the terminal constraints satisfy the constraint qualification conditions [13] *and* $\begin{bmatrix} \frac{\partial a}{\partial x} & diag\ (a) \\ \frac{\partial b}{\partial x} & 0 \end{bmatrix}$, it is full rank. Then in addition, to the necessary condition of optimality there exists Lagrangian multiplier vectors $\alpha \in \mathbb{R}^l, \beta \in \mathbb{R}^{l'}$, such that $\lambda(T) = S_x(x(T),T) + \alpha a_x(x(T),T) + \beta b_x(x(T),T)$ *where* $\alpha \ge 0, \alpha a(x(T),T) = 0$

2.3. **Problems with pure state inequality constraints [12]**

Consider the function $k(t,x)$ *where* $k = [0,T] \times \mathbb{R}^n$, and then the pure state constraints $k(t,x) \ge 0$ does not explicitly depend on u *and* x they can be controlled only indirectly. It is therefore, convenient to differentiate $k(t,x)$ with respect to time t as many times as required until it contains a control variable.

Let us for the moment define $k^i(t,x,u), for\ i = 1,2,\dots,p$, recursively as follows

$$\begin{cases} k^0(t,x,u) = k(t,x) \\ k^1(t,x,u) = \frac{d}{dx}k = k_x(t,x)g(t,x,u) + k_t(t,x) \\ k^2(t,x,u) = \frac{d}{dx}k^1 = k_x^1(t,x)g(t,x,u) + k_t^1(t,x) \\ \vdots \\ k^p(t,x,u) = \frac{d}{dx}k^{p-1} = k_x^{p-1}(t,x)g(t,x,u) + k_t^{p-1}(t,x) \end{cases} \longrightarrow (11)$$

Where subscripts denote partial derivatives, depending on the context we use a subscript such as i, to denote the i^{th} component of a vector

$$\text{If} \begin{cases} k_u^i(t,x,u) = 0, for\ 0 \le i \le p-1 \\ k_u^i(t,x,u) \neq 0, for\ i = p \end{cases} \longrightarrow (12)$$

Then the state constraint $k(t,x) \ge 0\ is\ of\ order\ p$

Generally, case of $k(t,x)$, the corresponding order p, for each component $k_i(t,x)\ of\ k(t,x)$, it is obtained from (11) and (12).

If the state constraints will of order of $p = 1$, and then it is easier to treat than the higher order case [10] with respect to thei^{th} , constraint $k_i(t,x) \ge 0$, a sub interval $(\tau_1,\tau_2) \subset [0,T]\ with\ \tau_1 < \tau_2$, it is called an interior interval of a tractor if $k_i(x(t),t) > 0, \forall\ t \in (\tau_1,\tau_2)$

An interval $[\tau_1,\tau_2]\ with\ \tau_1 < \tau_2$, it is called a **boundary interval** if $k_i(x(t),t) = 0\ for\ t \in [\tau_1,\tau_2]$

An instant τ_1, it is called an **entry time** if there is an interior interval ending at $t = \tau_1$, boundary interval starting at τ_1, correspondingly τ_2, it is called an **exist time** if a boundary interval ends at τ_2 and an interior interval starts at τ_2

If the trajectory x just touches the boundary at time $\tau, i.e. k\big(\tau, x(\tau_c)\big) = 0$, and if the trajectory x it is in the interior just before and after $\tau\ then\ \tau$, it is called a **contact time**.

Taken together, entry, exist and contact times are called **junction times**.

Assume that the following full rank conditions on any boundary interval $[\tau_1,\tau_2]\ is \begin{bmatrix} \frac{\partial k_1^{p1}}{\partial u} \\ \vdots \\ \frac{\partial k_{s}^{p_{s'}'}}{\partial u} \end{bmatrix}$ with full rank for all $t \in (\tau_1,\tau_2), where\ k_i^*(t) = 0\ for\ i = 1,2,\dots,s^{'} \le s\ and\ k_i^*(t) \ge 0\ for\ i = s^{'} + 1,,\dots,s$

That is the gradients of $k_i^{p_s}(t,x)$ with respect to u of the active constraints $k_i(x(t),t) = 0\ for\ i = 1,2,\dots,s^{'}$ they must be linearly independent along the optimal trajectory [3]

2.4. **Direct adjoint Approach**

In this approach, the Hamiltonian $\mathcal{H}\ and\ L$ the Lagrangian multipliers are defined as follows:

$\mathcal{H}(t,x,u,\lambda) = f(t,x,u) + \lambda g(t,x,u)$

$L(t,x,u,\lambda,\mu,v) = \mathcal{H}(t,x,u,\lambda) + \mu h(t,x,u) + vk(t,x)$ *where the vector* $\lambda \in \mathbb{R}^n(t)$ *and* $v \in \mathbb{R}^q(t)$

They are multipliers. This method derives its name from the fact that the mixed constraints $h(t,x,u) \geq 0$ as well as the pure state constraints $k(t,x) \geq 0$ they are directly adjoined to the Hamiltonian in order to form the Lagrangian.

Theorem – 2.5:

Let $\left(x^*(.), u^*(.)\right)$, it is an optimal pair for OCP over a fixed interval $[0,T]$ such that $u^*(.)$, it is right continuous with left hand limits and the constraint qualification condition of equation $\left(\frac{\partial h}{\partial u}, diag(h)\right)$ holds for every triple (t,x^*,u^*) *and* $t \in [0,T]$ *with* $u \in U\left(t,x^*(t)\right)$. Assume that $x^*(t)$, it has only finitely many junction times where $\lambda(.)$, they are continuous at junction time and then there exists a constant $\lambda_0(t) > 0$, a piece wise absolutely continuous co state trajectory $\lambda(.)$, mapping $[0,T]$ *into* $\mathbb{R}^n$, piece wise continuous multiplier functions $\mu(.)$*and* $v(.)$ mapping $[0,T]$ *into* $\mathbb{R}^s$ *and* $\mathbb{R}^q$repectively. A vector $\eta(\tau_i) \in \mathbb{R}^q$ *for each point* τ_i of discontinuity of $\lambda(.)$, and $\alpha \in \mathbb{R}^l, \beta \in \mathbb{R}^{l'}$ *and* $\gamma \in \mathbb{R}^q$, and such that$\left(\lambda_0, \lambda(t), \mu, v, \alpha, \beta, \eta(\tau_1), \ldots, \eta(\tau_i)\right) \neq 0, \forall\, t$ and the following conditions hold almost everywhere [10]

$$u^*(t) = \arg\max_{u\in U(t,x(t))} \mathcal{H}\left(t,x^*,u,\lambda_0,\lambda(.)\right)$$

$$L_u^*(t) = \mathcal{H}_u^*(t) + \mu h_u(t) = 0$$

$$\dot{\lambda} = -L_x^*(t)$$

$$\mu(t) \geq 0 \text{ and } \mu h^*(t,x,u) = 0$$

$$v \geq 0 \text{ and } vk^*(t,x) = 0$$

At the terminal time T, the following transversality conditions hold

$$\lambda(T^-) = \lambda_0 S_x^*(T) + \alpha a_x(T) + \beta b_x(T) + \gamma k_x^*(T) \text{ where } \alpha \geq 0, \gamma \geq 0 \text{ with } \alpha a(T) = \gamma k^*(T) = 0$$

For any time τ in a boundary interval and for any contact time τ, the co state trajectory λ, it may have a discontinuity given by the following jump conditions

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau)k_x^*(\tau);\ \mathcal{H}^*(\tau^-) = \mathcal{H}^*(\tau^+) - \eta(\tau)k_x^*(\tau) \text{ with } \eta(\tau) \geq 0 \text{ and } \eta(\tau)k_x^*(\tau) = 0$$

Where τ^+ *and* τ^- denote the left and right hand side limits respectively

Proposition – 2.6:

The adjoint function λ, it is continuous at a junction time τ *i.e.*$\eta(\tau) = 0$, if either conditions (a) or (b) holds:

a. The control u^*, it is continuous at τ *and* $\begin{bmatrix} \frac{\partial h^*(\tau)}{du} & diag\left(h^*(\tau)\right) & 0 \\ \frac{\partial k^{1*}(\tau)}{du} & 0 & diag\left(k^*(\tau)\right) \end{bmatrix} \longrightarrow (14)$
   It is full rank where $k^1(t,x,u)$as defined in equation (13)
b. The entry or exist is non tangential that is $k^{1*}(\tau^-) < 0$ *or* $k^{1*}(\tau^+) > 0$, and then $\lambda(t)$, it is continuous at time $t = \tau$ [10]

Definition – 2.7:

The Hamiltonian is said to be **regular** if along a given $x(t), \lambda(t), \eta(t)\ and\ \mathcal{H}\big(x(t), u,\ \lambda(t), \eta(t)\big)$, it has a unique maximum in $u, \forall\ t \in [0, T]$

Proposition – 2.8:

If the Hamiltonian is regular, which in this context means that the maximization of $\mathcal{H}$ with respect to u, it is unique and then u^*, it is continuous everywhere including the points on the boundary [9]

2.9. The indirect adjoining approach with complementary slackness (First order constrains)

The main idea behind this approach is "If the trajectory hits the boundary at time $\tau_1, i.e. k(x(\tau_1), \tau_1) = 0$, and then for it to remain on the boundary up to time τ_2 requires

$k^1\big(t, x^*(t), u^*(t)\big) = 0\ for\ t \in (\tau_1, \tau_2)\ where\ k^1(t, x, u)$, it may or may not depend explicitly on the control variables". This asserts that the phase velocity of a point moving along the trajectory is tangential to the boundary at time t. At the end point τ_2, we must have $k^{1*}(\tau_2^+) \geq 0$. Thus, one could formally impose the constraint $k^1(t, x, u) \geq 0\ whenever\ k(t, x) = 0$ in order to prevent the trajectory from violating the constraint $k(t, x) \geq 0$. Then the Hamiltonian and Lagrangian can be defined as follows:

$$\mathcal{H}^1(t, x, u, \lambda_0, \lambda^1) = \lambda_0 f(t, x, u) + \lambda^1 g(t, x, u)$$

$$L^1(t, x, u, \lambda_0, \lambda^1, \mu, v^1) = \mathcal{H}^1(t, x, u, \lambda_0, \lambda^1) + \mu h(t, x, u) + v^1 k^1(t, x, u)$$

Because the derivative of $k^1(t, x, u)\ of\ k(t, x)$ rather than $k(t, x)$ itself is adjoined to $\mathcal{H}$ in forming the Lagrangian, this approach is known as the indirect adjoining approach.

The control region is $U^1(t, x) = \{u \in \mathbb{R} |\ h(t, x, u) \geq 0, k^1(t, x, u) \geq 0\ if\ k(t, x) = 0\}$

The necessary conditions of optimality that are used as a procedure while applying the indirect adjoining approach are now stated as follows:

Theorem – 2.10:

Let $\big(x^*(.), u^*(.)\big)$, it is an optimal pair for OCP such that $x^*(.)$, it has only finitely many junction times and the strong constraint qualification condition of equation (14) holds. Then there exists a constant $\lambda_0 \geq 0$ a piece wise absolutely continuous co stat trajectory $\lambda^1(.)\ mapping\ [0, T]\ into\ \mathbb{R}^n$, piece wise continuous multiplier function $\mu(.)\ and\ v^1(.)\ mapping\ [0, T]\ into\ \mathbb{R}^s\ and\ \mathbb{R}^q$ respectively, a vector $\eta^1(\tau_i) \in \mathbb{R}^q\ \forall\ \tau_i$ of discontinuity of $\lambda^1(.)\ and\ \alpha \in \mathbb{R}^l, \beta \in \mathbb{R}^{l'}\ not\ all\ zero$, such that the following conditions hold almost everywhere [10]

$$u^*(t) = \arg\max_{u \in U(t, x(t))} \mathcal{H}^1\big(t, x^*, u, \lambda_0, \lambda^1(.)\big)$$

$\dot{\lambda}^1 = -L_x^{1*}(t);\ L_u^{1*}(t) = 0;\ \mu(t) \geq 0\ and\ \mu h^*(t, x, u) = 0$, and also

v_i^1, it is non – increasing on boundary intervals of $k_i(t, x), for\ i = 1, 2, \ldots, q\ \ with$

$v^1(t) \geq 0, \dot{v}^1 \leq 0\ and\ v^1 k^{1*}(t, x, u) = 0\ and\ also\ \frac{d\mathcal{H}^{*1}}{dt}(t) = \frac{dL^{*1}}{dt}(t) = L_t^{1*}(t)$

Whenever these derivatives exist, at the terminal time T the transversality conditions

$\lambda(T^-) = \lambda_0 S_x^*(T) + \alpha a_x(T) + \beta b_x(T) + \gamma k_x^*(T)\ where\ \alpha \geq 0, \gamma \geq 0\ with\ \alpha a(T) = \gamma k^*(T) = 0$, holds

At each entry or contact time, the co state trajectory λ^1, it may have a discontinuity of the form

$\lambda^1(\tau^-) = \lambda^1(\tau^+) + \eta^1(\tau)k_x^*(\tau);\ \mathcal{H}^{*1}(\tau^-) = \mathcal{H}^{*1}(\tau^+) + \eta^1(\tau)k_t^*(\tau)\ with\ \eta^1(\tau) \geq 0\ and\ \eta^1(\tau)k_t^*(\tau) = 0$

2.11. The indirect adjoining approach for higher order constraints

In this situation consider constraints of higher order i.e. $p \geq 2$, this means if $p = 1\ and\ k^1(t,x,u)$, it does not depend on the control value u, and then differentiate $k(t,x)$ with respect to time t as required until it contains a control variable u. Then such type of constraints are said to be indirect adjoint approach for higher order constraints. The Hamiltonian and Lagrangian of the indirect approach for the state constraint of order p are now

$\mathcal{H}^p(t,x^*,u,\lambda_0,\lambda^p) = \lambda_0 f(t,x,u) + \lambda^p g(t,x,u)$

$L^p(t,x^*,u,\lambda_0,\lambda^p,\mu,v^p) = \mathcal{H}^p(t,x^*,u,\lambda_0,\lambda^p) + \mu h(t,x,u) + v^p k^p(t,x,u)\ with\ k^p$, it is defined in (13). Then the control region $U^p(t,x)$, it is defined as follows

$U^p(t,x) = \{u \in \mathbb{R}^n \mid h(t,x,u) \geq 0, k^p(t,x,u) \geq 0\ if\ k(t,x) = 0\}$

Theorem – 2.12:

Let $(x^*(.),u^*(.))$, it is an optimal pair for OCP such that $x^*(.)$, it has only finitely many junction times and where constraint $k(t,x)$ of order p let the constraint qualification condition of equation (11) holds. Then there exists a constant $\lambda_0 \geq 0$ a piece wise absolutely continuous co stat trajectory $\lambda^p(.)\ mapping\ [0,T]\ into\ \mathbb{R}^n$, piece wise continuous multiplier function $\mu(.)\ and\ v^1(.)\ mapping\ [0,T]\ into\ \mathbb{R}^s\ and\ \mathbb{R}^q$ respectively, a vector $\eta^1(\tau_i) \in \mathbb{R}^q\ \forall\ \tau_i$ of discontinuity of $\lambda^p(.)\ and\ \alpha \in \mathbb{R}^l, \beta \in \mathbb{R}^{l'}\ not\ all\ zero$, such that the following conditions hold almost everywhere [10]

$u^*(t) = \arg\max_{u \in U(t,x(t))} \mathcal{H}^1(t,x^*,u,\lambda_0,\lambda^p(.))$

$\dot{\lambda}^p = -L_x^{p*}(t);\ L_u^{p*}(t) = 0;\ \mu(t) \geq 0\ and\ \mu h^*(t,x,u) = 0$, and also the multiplier function

v^p , it is differentiable $p - 1$times and $(v^p)^{p-1}$, it is of bounded variation

$(-1)(v^p)^r(t) \geq 0, for\ r = 0,1,\ldots,p, and\ v^p k^{p*}(t,x,u) = 0\ and\ also\ \frac{d\mathcal{H}^{*p}}{dt}(t) = \frac{dL^{*p}}{dt}(t) = L_t^{p*}(t)$

Whenever these derivatives exist, at the terminal time T the transversality conditions with λ^1 replaced by λ^p

At each entry or contact time, the co state trajectory λ^p, it may have discontinuity of the form

$$\lambda^p(\tau^-) = \lambda^p(\tau^+) + \sum_{r=1}^{p} \eta^r(\tau)(k^{r-1})_x^*(\tau) \longrightarrow (15)$$

$$\mathcal{H}^p(\tau^-) = \mathcal{H}^p(\tau^+) + \sum_{r=1}^{p} \eta^r(\tau)(k^{r-1})_x^*(\tau) \longrightarrow (16)$$

$$with\ \eta^r(\tau) \geq 0\ and\ \eta^r(\tau)k_t^*(\tau) = 0\ for\ r = 1,2,\dots,p \longrightarrow (17)$$

Proof:
By the condition of the maximum principle the necessary conditions for u^* with the state trajectory x^* to be optimal control for the problem is the same approach as first order state constraints (Indirect approach). Suppose that the constraint $k(t,x) \geq 0$ it is constraint of order p and since $k(t,x)$, it is derivable p times until it contains a control variable u. In the case of p order constraints, we need to define $k^p(t,x,u)$, as defined in (12). Then using p order constraints, we can for Lagrangian function as follows:

$L^p(t,x^*,u,\lambda_0,\lambda^p,\mu,v^p) = \mathcal{H}^p(t,x^*,u,\lambda_0,\lambda^p) + \mu h(t,x,u) + v^p k^p(t,x,u)$, where Hamiltonian is

$$\mathcal{H}^p(t,x^*,u,\lambda_0,\lambda^p) = \lambda_0 f(t,x,u) + \lambda^p g(t,x,u)$$

Since p, indicate order and assume that the function $g\ and\ k$ they are continuously differentiable with respect to all their argument up to order $(p-1)\ and\ p$ respectively and then the necessary condition of optimality as follows

$$u^*(t) = \arg\max_{u \in U(t,x(t))} \mathcal{H}^p\big(t,x^*,u,\lambda_0,\lambda^p(.)\big)$$

$$\dot{\lambda}^p = -L_x^{p*}(t);\ L_u^{p*}(t) = 0;\ \mu(t) \geq 0\ and\ \mu h^*(t,x,u) = 0$$

If the switching function of order p, the jump condition at entry times, the co state trajectory λ^p and Hamiltonian function may have a discontinuity of the form

$$\lambda^p(\tau^-) = \lambda^p(\tau^+) + \sum_{r=1}^{p} \eta^r(\tau)(k^{r-1})_x^*(\tau)$$

$$\mathcal{H}^p(\tau^-) = \mathcal{H}^p(\tau^+) + \sum_{r=1}^{p} \eta^r(\tau)(k^{r-1})_x^*(\tau)$$

$$with\ \eta^r(\tau) \geq 0\ and\ \eta^r(\tau)k_t^*(\tau) = 0\ for\ r = 1,2,\dots,p$$

2.13. The indirect adjoining approach with continuous adjoint functions [11]

In this case the adjoint function $\tilde{\lambda}$, it is continuous. The Hamiltonian $\mathcal{H}$ and the control region U respectively

$$\widetilde{\mathcal{H}} = \big(t,x^*,u,\lambda_0,\tilde{\lambda},\mu,\tilde{v}\big) = \lambda_0 f(t,x,u) + \tilde{\lambda} g(t,x,u) + \mu h(t,x,u) + \tilde{v} k^1(t,x,u)$$

$\widetilde{U}(t,x) = \{u \in \mathbb{R}^n \mid h(t,x,u) \geq 0, k^1(t,x,u) \geq 0\ if\ k(t,x) = 0\}$, and also

$$\cup\, \widetilde{U}(t,x) = \cup\{u \in \mathbb{R}^n \mid h(t,x,u) \geq 0, k^1(t,x,u) \geq 0\ if\ k(t,x) = 0\}$$

Theorem – 2.14:

Let $\big(x^*(.),u^*(.)\big)$, it is an optimal pair for OCP such that $x^*(.)$, it has only finitely, many junction times and where constraint $k(t,x)$ of order p let the constraint qualification condition of equation (14) holds. Then there exists a constant $\lambda_0 \geq 0$ continuous and a piece wise continuously differentiable adjoint function $\tilde{\lambda}(.): [0,T] \longrightarrow \mathbb{R}^n$ and multiplier function $\mu(.): [0,T] \longrightarrow$

$\mathbb{R}^s$ *and* $\tilde{v}(.) : [0,T] \longrightarrow \mathbb{R}^q$ respectively, such that the following conditions are satisfied whenever u, it is continuous

$$u^*(t) = \arg\max_{u \in U(t,x(t))} \widetilde{\mathcal{H}}(t, x^*, u, \lambda_0, \tilde{\lambda}, \mu, \tilde{v})$$

$\dot{\tilde{\lambda}} = -\widetilde{\mathcal{H}}_x(t);\ \widetilde{\mathcal{H}}_u(t) = 0;\ \frac{d\widetilde{\mathcal{H}}}{dt}(t) = \frac{d\widetilde{\mathcal{H}}}{dt}(t)$, and also the multiplier functions $\mu(.)$ *and* $\tilde{v}(.)$, they are continuous on intervals of continuity of $u^*(.)$, furthermore $\tilde{v}(.)$, it is non – increasing on $[0,T]$, continuous whenever $k_i^*(.)$, it is discontinuous that is when entry to or exit from the corresponding state constraint is non – tangential, and constant on intervals up on which $k_i^*(.) \geq 0$, at the terminal time T, the following transversality conditions hold

$$\tilde{\lambda}(t) = \lambda_0 S_x^*(T) + \alpha a_x(T) + \beta b_x(T)\ with\ \alpha \geq 0\ and\ \alpha a(T) = 0$$

Proof:
Suppose u^*, it is continuous and then Hamiltonian is regular along a given $x(t), \lambda(t), \eta(t)$ and then $\mathcal{H}(t, x(t), \lambda(t), \eta(t))$, it has a unique maximum in $u, \forall\ t \in [0,T]$, including the points on the boundary by proposition – 2.6
Therefore, the necessary conditions are hold since maximum principle is unique and using partial derivative we can obtain optimality conditions and the adjoint equations as follow

$$u^*(t) = \arg\max_{u \in U(t,x(t))} \widetilde{\mathcal{H}}(t, x^*, u, \lambda_0, \tilde{\lambda}, \mu, \tilde{v})$$

$$\dot{\tilde{\lambda}} = -\widetilde{\mathcal{H}}_x(t);\ \widetilde{\mathcal{H}}_u(t) = 0;\ \frac{d\widetilde{\mathcal{H}}}{dt}(t) = \frac{d\widetilde{\mathcal{H}}}{dt}(t)$$

Since the adjoint function is continuous and then at the terminal time T, the following transversality conditions hold as follow

$$\tilde{\lambda}(t) = \lambda_0 S_x^*(T) + \alpha a_x(T) + \beta b_x(T)\ with\ \alpha \geq 0\ and\ \alpha a(T) = 0$$

2.15: Existence Result [6]

We can review several different sets of optimality conditions for OCP. Since optimality conditions do not mean much in the absence of an optimal solution then we briefly provide some existence results for the problems. Our purpose in this paper is not to make a review of existence result. We choose to mention two characteristic

- The first result uses strong assumption such as boundedness of all admissible state and control paths
- The second result uses growth conditions on the sate and control variables [10]

2.16. The Growth condition [4]

If $f\ and\ g$, satisfy the following conditions for every bounded subset $X\ of\ \mathbb{R}^n$, and then there exists a constant c and summable functions d such that, for almost everywhere $t, \forall (x,u) \in dom\ f(t,x,u)$with $x \in X$, we have

$$\|g_x(t,x,u)\| \leq c\{|g(t,x,u)| + f(t,x,u)\} + d(t)\ and\ for\ all\ \xi\ and\ \psi$$

$$|\xi|(1 + \|g_u(t,x,u)\|) \leq c\{|g_u(t,x,u)| + f(t,x,u)\} + d(t)$$

Define the state dependent control region: $U(t,x) = \{u \in \mathbb{R}^n | h(t,x,u) \geq 0\} \subset \mathbb{R}^n$ and the set

$N(t,x) = \left\{f(t,x,u) + \gamma, \frac{g(t,x,u)}{\gamma} \leq 0, u \in U(t,x) \subset \mathbb{R}^n\right\}$

Lemma – 2.17 [2]

Let $U(y)$, it is an upper semi continuous set – valued mapping $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ with compact values. Then on any compact (and hence on any bounded) set of y, the values $U(y)$, they are uniformly bounded i.e. and for any compact set $K \in \mathbb{R}^m$, there exists a constant δ such that the set $U(y)$, it is contained in the ball $B(0,\delta)\ for\ any\ y \in K$

Corollary – 2.18 [2]

Suppose that the set $U(t,x)$, it is an upper semi continuous set – valued mapping $\mathbb{R}^{m+1} \longrightarrow \mathbb{R}^n$ with compact values. Then for any $T > 0$ and any bounded set $Q \subset \mathbb{R}^m$ there is an $R = R(T,Q)$ such that the inclusion $U(t,x) \subset B(0,\delta)\ holds\ for\ any\ t \in [0,T]\ and\ any\ x \in Q$

Theorem – 2.19

Consider the OCP where T , it is free to vary in the interval $[0,T]$, and assume that $f,g,h,k,S,a\ and\ b$ they are continuous in all their arguments at all points $(t,x,u) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^n$. Suppose that there exists an admissible solution pair and that the following conditions hold

a. $N(t,x)$, it is convex for all $(t,x) \in [0,T] \times \mathbb{R}^m$, and suppose further that
b. There exists $\delta > 0$ such that $\|u\| < \delta_1$, for all admissible pair $\big(x(t),u(t)\big)\ and\ t$, and
c. There exists $\delta_1 > 0$ such that $\|u\| < \delta_1, \forall\ u \in U(t,x)\ with\ \|x(t)\| < \delta$, and also
d. There exists an optimal triple $(T^*,x^*,u^*)\ with\ u^*(.)$, it is measurable

Proof:

Let $f_1(t,x,u) + \gamma_1, g_1(t,x,u)\ and\ f_2(t,x,u) + \gamma_2, g_2(t,x,u)\ \in N(t,x)$, and then for all any $0 \leq a \leq 1$

$$a\big(f_1(t,x,u) + \gamma_1, g_1(t,x,u)\big) + (1-a)\big(f_2(t,x,u) + \gamma_2, g_2(t,x,u)\big) =$$
$$af_1(t,x,u) + a\gamma_1, ag_1(t,x,u) + f_2(t,x,u) + \gamma_2, g_2(t,x,u) - af_2(t,x,u) - a\gamma_2 - ag_2(t,x,u)$$

Collect like terms together we get

$$\big(a(f_1(t,x,u) + \gamma_1) + (1-a)(f_2(t,x,u) + \gamma_2)\big), ag_1(t,x,u) + (1-a)g_2(t,x,u) =$$
$$\Big(a\big(f_1(t,x,u) - f_2(t,x,u)\big)\Big) + a(\gamma_1 - \gamma_2) + \gamma_2, ag_1(t,x,u) + (1-a)g_2(t,x,u)$$

$\Longrightarrow N(x,t)$, it is convex and also by the lemma 2.17 there exists $\delta > 0$ such that $\|x(t)\| < \delta$, for all admissible pair $\big(x(t),u(t)\big)\ and\ t$, and also there exists $\delta_1 > 0$ such that $\|u\| < \delta_1, \forall\ u \in U(t,x)\ with\ \|x(t)\| < \delta$

$\Longrightarrow$The triple $(T^*,x^*,u^*) \in$ the compact set $[0,T] \times B(0,\delta) \times B(0,\delta_1)$

$\Longrightarrow$The set of solutions $x(t)$ of OCP is uniformly bounded and continuous and the set of controls $u(t)$, it is uniformly bounded and hence the optimal triple $(T^*,x^*,u^*)\ with\ u^*(.)$, it is measurable

2.20. Sufficient conditions and uniqueness [1]

Theorem – 2.21

Let $(x^*(.), u^*(.))$, it is a feasible pair for the OCP with a fixed horizon time $T < \infty$. Then there exists a piece wise continuously differentiable function $\lambda: [0,T] \longrightarrow \mathbb{R}^n$ such that for every other feasible pair$(x(.), u(.))$, the following conditions hold

a. The maximum Hamiltonian
   $\mathcal{H}(t, x^*(t), u^*(t), \lambda(t)) - \mathcal{H}(t, x(t), u(t), \lambda(t)) \geq \dot{\lambda}(t)(x(t) - x^*(t)), \forall\, t \in [0,T]$
b. The jump conditions
   $\lambda(\tau^-) - \lambda(\tau^+)(x(\tau) - x^*(\tau)) \geq 0,\ \forall\, t \in [0,T]\ where\ \lambda$, it is discontinuous and
c. The transversality condition
   $\lambda(t)(x(T) - x^*(T)) \geq S(x^*(T), T)$, and then (x^*, u^*), it is optimal

Note:

This sufficient conditions and uniqueness do not use any concavity or convexity assumption [4], [7]

Theorem – 2.22 (Arrow type) [5]

Let $(x^*(.), u^*(.))$, it is a feasible pair for the OCP with a fixed horizon time $T < \infty$. Then there exists a piece wise continuously differentiable function $\lambda: [0,T] \longrightarrow \mathbb{R}^n$, piece wise continuous functions $\mu: [0,T] \longrightarrow \mathbb{R}^s\ and\ \eta: [0,T] \longrightarrow \mathbb{R}^q$ such that all necessary conditions hold and assume further that there exists $\alpha \in \mathbb{R}^l, \beta \in \mathbb{R}^{l'}$ such that the transversality conditions hold and assume that at all points τ_i of discontinuity of λ, and then there exists a $\eta(\tau_i) \in \mathbb{R}^n$ such that jump conditions hold. If the maximized Hamiltonian $\mathcal{H}^0(t, x, u, \lambda) = \max_{u \in U(t,x)} \mathcal{H}(t, x, u, \lambda)$, it is concave in $x, \forall (t, \lambda(t))\ and\ S(x, T)$ it is concave in $x\ and\ b(t, x)$, it is linear in $x\ then\ (x^*, u^*)$ it is an optimal pair [10]

Theorem – 2.23

A nonnegative linear combination of concave functions is also a concave function. That is, if $f^i: x \longrightarrow \mathbb{R}$, for $i = 1, 2, \dots, m$, they are concave functions on a convex subset $x \subset \mathbb{R}^n$, then

$f(x) = \sum_{i=1}^{m} \alpha^i f^i(x)\,, where\ \alpha^i \in \mathbb{R}^+$, it is also a concave function on $x \subset \mathbb{R}^n$

Proof:
This theorem implies for theorem 2.22
First recall the definition of the Hamiltonian, namely

$$\mathcal{H}(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u) = f(t, x, u) + \mu(t)(-g(t, x, u))$$

Since $g(t)$, it is convex in $(x^*(t), u^*(t)), \forall\, t \in [0,T]\ then - g(t)$, it is convex in $(x^*(t), u^*(t)) \forall\, t \in [0,T]$ by definition, moreover, because $\mu(t) \geq 0, \forall\, t \in [0,T]$

$\Longrightarrow\ \mathcal{H}(.)$, it is concave in$(x^*(t), u^*(t)), \forall\, t \in [0,T]$, by the theorem 2.23

Since $\mathcal{H}(.)$, it is a nonnegative linear combination of concave functions, thus, in either case, $\mathcal{H}(.)$ it is concave in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$

Finally, if $g(.)$, it is linear in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$ *then* $\lambda(t)$, it may be any sign and $\mathcal{H}(.)$ it is concave in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$

If $f(.)$, it is concave in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$, this should be clear since if $g(.)$, it is linear in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$, and then it is both concave and convex in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$

$\Longrightarrow\ \lambda(t)g(.)$, it is both concave and convex in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$ regardless of the sign in $\lambda(t)$

$\Longrightarrow\ g(.)$, it is linear in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$, and $f(.)$, it is concave in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$, and then $\mathcal{H}(.)$, it is concave in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$

Since it is a nonnegative linear combination of concave functions, in this instance we may also conclude that a solution of the necessary conditions of OCP is a solution to the OCP by theorem – 2.22, since $\mathcal{H}^0$, it is concave in $(x^*(t), u^*(t)), \forall\, t \in [0, T]$

Corollary – 2.24

If the assumption of theorem – 2.21 are satisfied and if theorem 2.22 holds with strict inequality for $x(t) \neq x^*(t)$, and then the optimal state trajectory $x^*(t)$, it is uniquely determined [9]

Corollary – 2.25

If the assumption of theorem – 2.22 is satisfied and if $\mathcal{H}^0(.)$ holds and it is strictly concave in x, and then the optimal state trajectory$x^*(t)$, it is unique

Note:
Corollary 2.24 and 2.25 may not be true the uniqueness of the optimal control u^*, in the case of $T = \infty$, and then the theorem – 2.23, it must be modified as follows [12]

Theorem – 2.26

If $T = \infty$, and then the theorems 2.21 and 2.22 remain valid if the transversality condition

$\lambda(T^-) = S_x^*(T) + \alpha a_x(T) + \beta b_x(T) + \gamma k_x^*(T)$ *where* $\alpha \geq 0, \gamma \geq 0$ *and* $\alpha a(T) = \gamma k^*(T) = 0$, and

$\lambda(T)\big(x(T) - x^*(T)\big) \geq S(x(T), T) - S(x^*(T), T)$, they are replaced by the following limiting transversality condition

$\lim\limits_{t \to \infty} \lambda(T)\big(x(T) - x^*(T)\big) \geq 0$, for every other feasible state trajectory $x(.)$

Proof:
Let $(x^*(t), u^*(t))$, it is any admissible pair, by hypothesis $L(.) \in C^{(1)}$ it is a concave function of $(x, u)\ \forall\, t \in [0, \infty]$

$$\Longrightarrow L(t, x^*, u^*, \lambda, \mu) = L(t, x, u, \lambda, \mu) + L_x(t, x, u, \lambda, \mu)(x^* - x) + L_u(t, x, u, \lambda, \mu)(u^* - u), \forall\, t \in [0, \infty]$$

Using the fact that $L_u(t,x,u,\lambda,\mu)=0$, and then integrating both sides of the resulting reduced inequality over the interval$[0,\infty]$, and again using the definition of $L(.)$ *and* $F(.)$, yields

$$F(x^*(.),u^*(.)) \geq \begin{cases} F(x(.),u(.)) + \\ \int_0^\infty \lambda\left(g(t,x,u)-g(t,x^*,u^*)\right)dt + \\ \int_0^\infty \mu\left(h(t,x,u)-h(t,x^*,u^*)\right)dt + \\ \int_0^\infty L_x(t,x,u,\lambda,\mu)(x^*-x)\,dt \end{cases} \longrightarrow \quad (18)$$

By admissibility

$\dot{x}(t)=g(t,x,u)$ *and* $\dot{x}^*(t)=g(t,x^*,u^*), \forall\, t\in[0,\infty]$ *while* $\dot{\lambda}=-L_x(t,x,u,\lambda,\mu), \forall\, t\in[0,\infty]$

Substituting these above three results in (18) we get

$$F(x^*(.),u^*(.)) \geq \begin{cases} F(x(.),u(.)) + \\ \int_0^\infty \lambda\left(\dot{x}(t)-\dot{x}^*(t)\right)dt + \\ \int_0^\infty \mu\left(h(t,x,u)-h(t,x^*,u^*)\right)dt + \\ \int_0^\infty \dot{\lambda}(x^*-x)\,dt \end{cases} \longrightarrow \quad (19)$$

Since $\mu(t)h(t,x,u)=0\ for\ \mu(t)\geq 0;\ \mu(t)h(t,x^*,u^*)=0\ for\ \mu(t)\geq 0$, and then we have

$$\int_0^\infty \mu\left(h(t,x,u)-h(t,x^*,u^*)\right)dt \leq 0 \longrightarrow \quad (20)$$

Using (20) in (19) we will reduce the form of (19) as follows

$$F(x^*(.),u^*(.)) \geq F(x(.),u(.)) + \int_0^\infty \lambda\left(\left(\dot{x}(t)-\dot{x}^*(t)\right)+\dot{\lambda}\left(x(t)-x^*(t)\right)\right)dt$$

$$\Longrightarrow F(x^*(.),u^*(.)) \geq F(x(.),u(.)) + \int_0^\infty \frac{d}{dt}\left[\lambda(t)\left(x(t)-x^*(t)\right)\right]dt$$

$$\Longrightarrow F(x^*(.),u^*(.)) \geq F(x(.),u(.)) + \lim_{t\to\infty}\left[\lambda(t)\left(x(t)-x^*(t)\right)\right]-\left[\lambda(0)\left(x(0)-x^*(0)\right)\right]$$

Since by admissibility we have $x(0)=x_0$ *and* $x^*(0)=x_0 \Longrightarrow \left(x(0)-x^*(0)\right)=0$

$\Longrightarrow F(x^*(.),u^*(.)) \geq F(x(.),u(.)) + \lim_{t\to\infty}\left[\lambda(t)\left(x(t)-x^*(t)\right)\right]$, and also for every admissible control path $u(t)$, $\lim_{t\to\infty}\left[\lambda(t)\left(x(t)-x^*(t)\right)\right]\geq 0$ *where* $x(t)$, it is the time path of the state variable corresponding to $u(t)$, and then it follows that

$F(x^*(.),u^*(.)) \geq F(x(.),u(.))$, for all admissible functions $(x^*(.),u^*(.))$

If $L(.)$, it is a strictly concave function of $(x^*(.),u^*(.))\ \forall\, t\in[0,\infty]$, and then the inequality becomes strict if either $x^*(t)\neq x(t)$ *or* $u^*(t)\neq u(t)$ *for some* $t\in[0,\infty)$

$$\Longrightarrow F(x^*(.),u^*(.)) \geq F(x(.),u(.))$$

This shows that any admissible pair of functions $(x^*(.),u^*(.))$, which are not identically equal to$(x(.),u(.))$, they are **sub optimal** [5]

2.28. Solving problem by using theorems 2.5, 2.10 and 2.12

Example:

Consider $\max \int_0^3 -x dt$

Subject to $\dot{x} = u, x \geq 0$

$u + 1 \geq; 1 - u \geq 0$

With $x(0) = 1 \; and \; x(3) = 1$

Solution:

The Hamiltonian is $\mathcal{H} = -x + \lambda u$

$\Longrightarrow$The optimal control to be $u^* = bang\,[-1, 1; \lambda] \; when \; x \geq 0$

And which optimal control on the state constraint bounder is $u^* = bang\,[1, 1; \lambda] \; when \; x = 0$

The bounder conditions $x(0) = 1 \; and \; x(3) = 1$

$$\Longrightarrow u^*(t) = \begin{cases} -1 \; for \; t \in [0, 1) \\ 0 \; for \; t \in [1, 2] \\ 1 \; for \; t \in (2, 3] \end{cases} \quad and \quad x^*(t) = \begin{cases} 1 - t \; for \; t \in [0, 1) \\ 0 \; for \; t \in [1, 2] \\ t - 2 \; for \; t \in (2, 3] \end{cases}$$

First we apply the direct adjoint approach and the Lagrangian form as

$$L = \mathcal{H} + \mu_1(u + 1) + \mu_2(1 - u) + vx$$

The necessary conditions of theorem 2.5 are $L_u = \lambda + \mu_1 - \mu_2 = 0 \; and \; \dot{\lambda} = -L_x = 1 - v$, but $\mu_1 \geq 0 \; and \; \mu_1(u + 1) = 0; \; \mu_2 \geq 0 \; and \; \mu_2(1 - u); v \geq 0, vx = 0 \; and \; \dot{v} \leq 0 \; and \; also \; \lambda(3) = \beta, where \; \beta \in \mathbb{R}$

The enters of the boundary of $x = 0$, in a non – tangential way at time $\tau_1 = 0$

Since $k^1(1^-) \leq 0 \; and \; also \; at \; time \; \tau_2 = 2$, it leaves this bounder non tangential

Since $k^1(2^+) \geq 0$ $\Longrightarrow$By the proposition 2.8 λ, it is continuous at time $t = 1 \; and \; t = 2$ as well as in $[0, 1) \; and \; (2, 3]$ where the stat constraint is not active.

Consider the boundary interval $[1, 2]$, here $u = 0 \Longrightarrow \mu_1 = \mu_2 = 0$, and then form

$L_u = \lambda + \mu_1 - \mu_2 = 0 \; and \; \lambda = 0$

Thus λ, it is also continuous in $(1, 2)$, furthermore, since $\lambda = 0$ and then $\dot{\lambda} = -L_x = 1 - v$, becomes

$-L_x = 1 - v = 0 \Longrightarrow v = 1$

$\Longrightarrow$All multipliers are uniquely determined in $[1, 2]$.

In $[0, 1)$, we have $x > 0 \; and \; v = 0 \; then \; \lambda = t - 1 \; because \; of \; \dot{\lambda} = 1 \; and \; \lambda(1) = 0$

Similarly in $(2, 3]$, we have $x > 0 \; and \; v = 0 \; then \; \lambda = t - 2 \; because \; of \; \dot{\lambda} = 1 \; and \; \lambda(2) = 0$

Determine $\mu_1 and \; \mu_2, from \; L_u = \lambda + \mu_1 - \mu_2 = 0$, and $\mu_1 \geq 0 \; and \; \mu_1(u + 1) = 0; \; \mu_2 \geq 0 \; and \; \mu_2(1 - u)$

In $[0, 1)$, we have $\lambda = t - 1 \; and \; u = -1 \; then \; \mu_2 = 0 \; and \; \mu_1 = 1 - t$

In $(2, 3]$, we have $\lambda = t - 2 \; and \; u = 1 \; then \; \mu_1 = 0 \; and \; \mu_2 = t - 2$

In the indirect adjoining approach the Hamiltonian $\mathcal{H}^1$ and Lagrangian L^1 they are

$\mathcal{H}^1 = -x + \lambda u$; $L^1 = \mathcal{H}^1 + \mu_1(u + 1) + \mu_2(1 - u) + v^1 u$

The necessary conditions of theorem 2.10 are

$L_u^1 = \lambda + \mu_1 - \mu_2 + v^1 = 0 \; and \; \dot{\lambda} = -L_x = 1 \; where \; \mu_1, \mu_2 \; and \; v^1$, satisfy the complementary slackness conditions they are

$\mu_1 \geq 0 \; and \; \mu_1(u + 1) = 0; \; \mu_2 \geq 0 \; and \; \mu_2(1 - u) = 0; \; v^1 \geq 0, v^1 x = 0 \; and \; \dot{v}^1 \leq 0$

Since $x^*(t)$ enters the boundary zero at $t = 1$ there are no jumps in interval $(1, 2]$, and the solutions for $\lambda^1(t)$

It is $\lambda^1(t) = t - 2 \; for \; t \in (1, 2]$

$\Longrightarrow \mathcal{H}^1(1^+) = -x^*(1^+) + \lambda^1(1^+)u^*(1^+) = 0 \; and \; also \; \mathcal{H}^1(1^-) = -x^*(1^-) + \lambda^1(1^-)u^*(1^-) = -\lambda^1(1^-)$

By the equations $\mathcal{H}^1(1^+) \; and \; \mathcal{H}^1(1^-)$, we get $\lambda^1(1^-) = 0$

Then the value of the jump condition is $\eta^1(1) = \lambda^1(1^-) - \lambda^1(1^+) = 1 \geq 0$
In time interval $[0,1)$,$\mu_2 = 0\ since\ u^* = -1\ and\ v^1 = 0\ because\ x > 0\ for\ t \in [0,1)$
$\Rightarrow \frac{\partial L}{\partial x} = \lambda + \mu_1 - \mu_2 + v = 0\ then\ \lambda^1 + \mu_1 = 0\ since\ \mu_2 = 0\ and\ v = 0\ for\ t \in [0,1)$
$\Rightarrow \mu_1(t) = -\lambda^1(t) = 2 - t\ for\ t \in [0,1)\ with\ u = -1$
At $t = 1$, we have $x(1) = 0 \Rightarrow$optimal control $u^*(1) = 0$
Assume that continue to use the control $u^*(t) = 0$, in the interval $[1,2]\ then\ x(t) = 0, for\ t \in [1,2]$
Since $\lambda^1(t) < -0\ for\ for\ t \in [1,2]\ then\ \ u^*(1) = 0$, on the same interval and then
$\mu_1 = \mu_2 = 0 for\ t \in [1,2]$, but we can obtain $v^1(t) = -\lambda^1(t)0 for\ t \in [1,2]$
$\Rightarrow$The adjoint function λ, it is continuous everywhere, v it is constant in $[0,1)\ and\ (2,3]$ where the state constraint is not active and v, it is continuous at $t = 1,2\ where\ k^1(t,x,u) = \dot{x} = u$, it is discontinuous. The adjoint function λ, it is continuous, since the entry to and the exit from the state constraint is non – tangential
Hence complete the result

Conclusion:
The proof of direct, indirect (First and higher order) adjoint approach with complementary slackness, any admissible pair of functions $\left(x^*(.), u^*(.)\right)$, which are not identically equal to$\left(x(.), u(.)\right)$, they are sub optimal, $\mathcal{H}^0$, it is concave in $\left(x^*(t), u^*(t)\right), \forall\ t \in [0,T], \mathcal{H}(.)$, it is a concave function of $\left(x^*(t), u^*(t)\right), \forall\ t \in [0,T], \left(x^*(t), u^*(t)\right)$, it is an optimal pair and also $x(t)$of OCP is uniformly continuous and bounded, $u(t)$, it is uniformly bounded. For the further research we can solve OCP in penalty method, non linear programming method and dynamic programming method

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