FIXED POINT THEOREM IN A COMPLETE MENGER SPACE USING IMPLICIT RELATION

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ABSTRACT: *In this paper, a common fixed point theorem of six self-maps is proved using semi-compatiblity, weak compatiblity, reciprocal continuity and implicit relation in Integral setting in a Menger space.*

KEYWORDS: Semi-compatible, Weak compatible, Reciprocal continuity,Implicit relation in Integral setting, Menger Space.

1. INTRODUCTION

The Probabilistic Metric Space (or Statistical Metric Space) was defined by Menger [5] in 1944, as a generalization of metric space. Then Schweizer and Sklar [7] gave some basic results in this space. Some mathematicians observed that condition of contraction in metric space may be translated into PM-Space with minimum norm. Sehgal and Bharucha [8] gave a generalization of Banach contraction principle in Menger space. Some basic definitions and theorems in Menger space which are used for proving the main result are as follows.

Definition 1.1 [7] "Let $\Delta : [0,1] \times [0,1] \to [0,1]$ be a mapping. Then Δ is said to be a triangular-norm (briefly, t-norm) if for all $\alpha,\ \beta,\ \gamma \in [0,1]$,

(i) $\Delta(\alpha, 1) = \alpha,\ \Delta(0,0) = 0;$

(ii) $\Delta(\alpha,\ \beta) = \Delta(\beta, \alpha);$

(iii) $\Delta(\alpha,\ \beta) \geq \Delta(\gamma,\ \delta)$ for $\alpha \geq \gamma,\ \beta \geq\ \delta;$

(iv) $\Delta(\Delta(\alpha,\ \beta),\ \gamma) = \Delta\big(\alpha,\ \Delta(\beta,\ \gamma)\big)$."

Example 1.2 [7] "The four basic t-norms are as follows:

(i) The minimum t-norm: $\Delta_M(\alpha,\ \beta) = \min\{\alpha,\ \beta\}$.

(ii) The product t-norm: $\Delta_{\ p}(\alpha,\ \beta) = \alpha\beta$.

(iii) The Lukasiewicz t-norm: $V_L(\alpha, \beta) = \min\{\alpha + \beta - 1, 0\}$.

(iv) The weakest t-norm, the drastic product:

$$\Delta_D(\alpha, \beta) = \begin{cases} \min\{\alpha, \beta\} & \text{if } \max\{\alpha, \beta\} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following ordering in the above stated norms:

$\Delta_D < \Delta_L < \Delta_P < \Delta_M$."

Definition 1.3 [7] "A mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}^+$ is a distribution function if it is left continuous and non-decreasing with inf $\mathcal{F}(x) = 0$ and sup $\mathcal{F}(x) = 1$ for all real x."

We shall denote the set of all distribution functions by $\mathcal{L}$ whereas $\mathcal{H}(t)$ be the Heaviside distribution function defined as

$$\mathcal{H}(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.4 [6] "The ordered pair $(\mathcal{K}, \mathcal{F})$ is called a PM space if $\mathcal{K}$ be a non-empty set and $\mathcal{F}: \mathcal{K} \times \mathcal{K} \to \mathcal{L}$ be a mapping satisfying:

(p_1) $\mathcal{F}_{x,y}(t) = 1$ for all $t > 0$, if and only if $x = y$;

(p_2) $\mathcal{F}_{x,y}(0) = 0$;

(p_3) $\mathcal{F}_{x,y}(t) = \mathcal{F}_{y,x}(t)$;

(p_4) $\mathcal{F}_{x,y}(t) = 1$ and $\mathcal{F}_{y,z}(s) = 1$, then $\mathcal{F}_{x,z}(t + s) = 1$,

for all x, y, z in $\mathcal{K}$ and t, s ≥ 0 .

Every metric space can always be realized as a probabilistic metric space by putting the relation $\mathcal{F}_{x,y}(t) = \mathcal{H}(t - d(x,y))$ for all x, y in $\mathcal{K}$."

Definition 1.5 [6] "The ordered triplet $(\mathcal{K}, \mathcal{F}, \Delta)$ is called a Menger space if $(\mathcal{K}, \mathcal{F})$ is a probabilistic metric space, Δ is a t-norm and satisfies for all x, y, z in $\mathcal{K}$ and t, s ≥ 0,

(p_5) $\mathcal{F}_{x,z}(t + s) \geq \Delta\left(\mathcal{F}_{x,y}(t), \mathcal{F}_{y,z}(s)\right)$."

Definition 1.6 [6] "A sequence $\{x_n\}$ in a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ is said to be:

(i) Cauchy sequence in $\mathcal{K}$ if for every $\epsilon > 0$ and $\lambda > 0$, we can find a positive integer $N_{\epsilon,\lambda}$ satisfying $\mathcal{F}_{x_n,x_m}(\epsilon) > 1 - \lambda$, for all $n, m \geq N_{\epsilon,\lambda}$.

(ii) Convergent at a point x $\in \mathcal{K}$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ satisfying $\mathcal{F}_{x_n,x}(\epsilon) > 1 - \lambda$, for all $n \geq N_{\epsilon,\lambda}$."

The space $\mathcal{K}$ is said to be complete if every Cauchy sequence is convergent in $\mathcal{K}$.

Definition 1.7 [6] "Let S and T be two self-mappings of a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$. Then S and T are said to be compatible if $\lim_{n\to\infty} \mathcal{F}_{STx_n,TSx_n}(t) = 1$ for all $t > 0$ where $\{x_n\}$ is a sequence in $\mathcal{K}$ satisfying

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$, where $u \in \mathcal{K}$."

Definition 1.8 [10] "Two self-mappings A and S of a non-empty set $\mathcal{K}$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points i.e. if Az = Sz for some z ∈ $\mathcal{K}$, then ASz = SAz."

Theorem 1.9 [10] "If two self-mappings A and S of a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ are compatible, then they are weakly compatible."

Definition 1.10 [2] "Let S and T be two self-mappings of a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$. Then S and T are said to be compatible of type (A) if we can find a sequence $\{x_n\}$ in $\mathcal{K}$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$, where $u \in \mathcal{K}$ and $\lim_{n\to\infty} \mathcal{F}_{STx_n, TTx_n}(t) = 1$ and $\lim_{n\to\infty} \mathcal{F}_{TSx_n, SSx_n}(t) = 1$ for all $t > 0$."

Definition 1.11 [2] "Let S and T be two self-mappings of a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$. Then S and T are said to be compatible of type (β) if we can find a sequence $\{x_n\}$ in $\mathcal{K}$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$, where $u \in \mathcal{K}$ and $\lim_{n\to\infty} \mathcal{F}_{SSx_n, TTx_n}(t) = 1$ for all $t > 0$."

Definition 1.12 [1] "Two self-maps S and T of a set $\mathcal{K}$ are occasionally weakly compatible maps (shortly owc) if and only if we can find a point x in $\mathcal{K}$ satisfying $Sx = Tx$ and $STx = TSx$."

Theorem 1.13 [3] "Let S and T be compatible maps of type (A) in a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ and $Sx_n, Tx_n \to u$ for some u in $\mathcal{K}$. Then

(i) $TSx_n \to Su$ if S is continuous.

(ii) STu = TSu and Su = Tu if S and T are continuous."

Theorem 1.14 [11] "Let $(\mathcal{K}, \mathcal{F}, \Delta)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that $\mathcal{F}_{x_{n+1}, x_n}(kt) \geq \mathcal{F}_{x_n, x_{n-1}}(t)$ for all x, y in $\mathcal{K}$ and t > 0, then $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$."

Theorem 1.15 [10] "Let $(\mathcal{K}, \mathcal{F}, \Delta)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that $\mathcal{F}_{x,y}(kt) \geq \mathcal{F}_{x,y}(t)$ for all x, y in $\mathcal{K}$ and t > 0, then x = y."

Theorem 1.16 [10] "In a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ if Δ(a, a) ≥ a, for all a ∈ [0, 1], then Δ (a,b) = Min{a, b} for a, b ∈ [0, 1]."

Definition 1.17 [15] "Let S and T be two self-mappings on a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$. Then S and T are called reciprocally continuous if

$\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$,

whenever $\{x_n\}$ is a sequence in $\mathcal{K}$ satisfying $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$, $z \in \mathcal{K}$."

Definition 1.18 [14] "Let S and T be two self-mappings of a Mengerspace $(\mathcal{K}, \mathcal{F}, \Delta)$with continuous t-norm Δ. Then S and T are called semi compatible if

$\lim_{n\to\infty} \mathcal{F}_{STx_n,Su}(t) = 1,$

whenever$\{x_n\}$ in $\mathcal{K}$ satisfies$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$,where $u \in \mathcal{K}$, $t > 0$."

Theorem 1.19 [16] "If self-mappings A and S of a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ are semi-compatible then they are weak compatible."

Theorem 1.20 [16] "Let S and T be two self-maps on a Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ with $\Delta(a,a) \geq a$, for all $a \in [0,1]$ and T is continuous. Then (S,T) is semi-compatible if and only if (S,T) is compatible."

Definition 1.21 [15] A Class of Implicit Relation. "Let $\mathbf{\Phi}$ be the set of all real continuous functions $\varphi : (\mathbb{R}^+)^4 \to \mathbb{R}$, non-decreasing in the first argument with the property:

(a) for u, v ≥ 0, $\phi(u,v,v,u) \geq 0$ or $\phi(u,v,u,v) \geq 0$ implies that u ≥ v;

(b)$\phi(u,u,1,1) \geq 0$ implies u ≥ 1."

Branciari proved the following theorem:

Theorem 1.22 [13] "Let (X, d) be a complete metric space. Suppose f : X → X be a mapping such that for each x, y ∈ X and c ∈ [0,1),

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt;$$

where $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is a summable (with finite integral) on each compact subset of$\mathbb{R}^+$, non-negative and such that for each $\epsilon > 0, \int_0^{\epsilon} \varphi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that for each $x \in X, \lim_{n\to\infty} f^n x = z$."

Definition 1.23 Implicit Relation in Integral Setting**:** Let $\mathbf{\Phi}$ be the set of all real continuous functions $\varphi : (\mathbb{R}^+)^4 \to \mathbb{R}$, non-decreasing in the first argument with the property:

(a) For u, v ≥ 0,$\int_0^{\phi(u,v,v,u)} \psi(t)dt \geq 0$ or $\int_0^{\phi(u,v,u,v)} \psi(t)dt \geq 0$ implies that u ≥ v.

(b)$\int_0^{\phi(u,u,1,1)} \psi(t)dt \geq 0$ implies u ≥ 1,

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$is a Lebesgue integrable mapping which is a summable (with finite integral) on each compact subset of $\mathbb{R}^+$, non-negative and such that for each

$\in > 0$, $\int_0^{\in} \psi(t)dt > 0$.

Theorem 1.24 [12] "Let $(\mathcal{K}, \mathcal{F}, \Delta)$be a Menger space. If there exists a constant k ∈ (0, 1) such that $\int_0^{\mathcal{F}_{x,y}(kt)} \psi(t)dt \geq \int_0^{\mathcal{F}_{x,y}(t)} \psi(t)dt$for all t > 0 with fixed x, y ∈$\mathcal{K}$, where ψ: [0, 1) → [0, 1) is a non-negative summable Lebesgue integrable function such that $\int_\varepsilon^1 \psi(t)dt > 0$ for each ε ∈ [0, 1), then x = y."

2. Main Result

Theorem 2.1 Let A, B, S, T, I and J be self-mappings of a complete Menger space $(\mathcal{K}, \mathcal{F}, \Delta)$ such that

(i) AB($\mathcal{K}$) ⊂ J($\mathcal{K}$) and ST($\mathcal{K}$) ⊂ I($\mathcal{K}$);

(ii) the pair (AB , I) is semi-compatible and (ST , J) is weak compatible;

(iii) the pair (AB , I) or (ST, J) is reciprocally continuous;

(iv) for some φ ∈ Φ, there exists k ∈ (0, 1) such that for all x, y ∈$\mathcal{K}$ and t > 0 ,

$$\int_0^{\phi(\mathcal{F}_{ABx,STy}(kt),\ \mathcal{F}_{Ix,Jy}(t),\ \mathcal{F}_{ABx,Ix}(t),\ \mathcal{F}_{STy,Jy}(kt))} \psi(t)dt \geq 0, \qquad (2.1)$$

$$\int_0^{\phi(\mathcal{F}_{ABx,STy}(kt),\ \mathcal{F}_{Ix,Jy}(t),\ \mathcal{F}_{ABx,Ix}(kt),\ \mathcal{F}_{STy,Jy}(t))} \psi(t)dt \geq 0. \qquad (2.2)$$

Then AB, ST, I and J have a unique common fixed point .

Furthermore, if the pairs (A, B), (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof. Let x_0∈ $\mathcal{K}$. Since AB($\mathcal{K}$) ⊂ J($\mathcal{K}$) and ST($\mathcal{K}$) ⊂ I($\mathcal{K}$),

there exist x_1, x_2∈ $\mathcal{K}$ such that ABx_0= Mx_1= $y_0$and STx_1= Lx_2= y_1.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{K}$ such that

ABx_{2n}= Jx_{2n+1}= $y_{2n+1}$and STx_{2n+1}= Ix_{2n+2}= y_{2n+2} for n = 0, 1, 2,...

Now putting x = x_{2n} , y = x_{2n+1}in inequality (2.1) , we obtain

$$\int_0^{\phi(\mathcal{F}_{ABx_{2n},STx_{2n+1}}(kt),\ \mathcal{F}_{Ix_{2n},Jx_{2n+1}}(t),\ \mathcal{F}_{ABx_{2n},Ix_{2n}}(t),\ \mathcal{F}_{STx_{2n+1},Jx_{2n+1}}(kt))} \psi(t)dt \geq 0. \qquad (2.3)$$

That is,

$$\int_0^{\phi(\mathcal{F}_{y_{2n+1},y_{2n+2}}(kt),\ \mathcal{F}_{y_{2n},y_{2n+1}}(t)),\ \mathcal{F}_{y_{2n},y_{2n+1}}(t)),\ \mathcal{F}_{y_{2n+2},y_{2n+1}}(kt))} \psi(t)dt \geq 0. \qquad (2.4)$$

Using (a) of Definition 1.21, we get

$\mathcal{F}_{y_{2n+2},y_{2n+1}}(kt) \geq \mathcal{F}_{y_{2n+1},y_{2n}}(t)$. (2.5)

Analogously, putting $x = x_{2n+2}$, $y = x_{2n+1}$ in (2.2), we have

$$\int_0^{\phi(\mathcal{F}_{ABx_{2n+2},STx_{2n+1}}(kt),\mathcal{F}_{Ix_{2n+2},Jx_{2n+1}}(t),\ \mathcal{F}_{ABx_{2n+2},Ix_{2n+2}}(kt),\mathcal{F}_{STx_{2n+1},Jx_{2n+1}}(t))} \psi(t)dt \geq 0.$$

Using (a) of Definition 1.24, we get

$\mathcal{F}_{y_{2n+3},y_{2n+2}}(kt) \geq \mathcal{F}_{y_{2n+2},y_{2n+1}}(t)$.(2.6)

Thus, from (2.5) and (2.6), for any n and t, we have

$\mathcal{F}_{y_n,y_{n+1}}(kt) \geq \mathcal{F}_{y_{n-1},y_n}(t)$. (2.7)

Hence by Theorem 1.14, $\{y_n\}$ is a Cauchy sequence in $\mathcal{K}$ which is complete. Therefore $\{y_n\}$ converges to p $\in \mathcal{K}$. The sequences $\{ABx_{2n}\}$, $\{STx_{2n+1}\}$, $\{Ix_{2n}\}$

And $\{Jx_{2n+1}\}$, being subsequences of $\{y_n\}$ also converge to p , that is

$$\{ABx_{2n}\} \to p,\ \{STx_{2n+1}\} \to p, \quad (2.8)$$

$$\{Ix_{2n}\} \to p,\ \{Jx_{2n+1}\} \to p. \quad (2.9)$$

The reciprocal continuity of the pair (AB, I) gives

$ABIx_{2n} \to ABp$ and $IABx_{2n} \to Ip$.

The semi-compatibility of the pair (AB, I)gives $\lim_{n\to\infty} ABIx_{2n} = Ip$.

From the uniqueness of the limit in a Menger metric space, we obtain that

ABp = Ip (2.10)

Step1. By putting x = p, $y = x_{2n+1}$ in (2.1), we obtain

$$\int_0^{\phi(\mathcal{F}_{ABp,STx_{2n+1}}(kt),\ \mathcal{F}_{Ip,Jx_{2n+1}}(t),\ \mathcal{F}_{ABp,Ip}(t),\ \mathcal{F}_{STx_{2n+1},Jx_{2n+1}}(kt))} \psi(t)dt \geq 0.$$

Letting $n \to \infty$ and using (2.8), (2.9) and (2.10), we get

$$\int_0^{\phi(\mathcal{F}_{Ip,p}(kt),\ \mathcal{F}_{Ip,p}(t),\ \mathcal{F}_{Ip,Ip}(t),\ \mathcal{F}_{p,p}(kt))} \psi(t)dt \geq 0 .$$

As φ is non-decreasing in first argument, we have

$$\int_0^{\phi(\mathcal{F}_{Ip,p}(t),\ \mathcal{F}_{Ip,p}(t),\ 1,\ 1)} \psi(t)dt \geq 0 .$$

Using (b) of Definition 1.21, we have $\mathcal{F}_{Ip,p}(t) \geq 1$ for all t > 0,

which gives $\mathcal{F}_{Ip,p}(t) = 1$, that is Ip = p = ABp.(2.11)

Step 2. As $AB(\mathcal{K}) \subset J(\mathcal{K})$, there exists $u \in \mathcal{K}$ such that AB p = I p = p = J u.

Putting $x = x_{2n}$, y = u in (2.1) we obtain that

$$\int_0^{\phi(\mathcal{F}_{ABx_{2n},STu}(kt),\ \mathcal{F}_{Ix_{2n},Ju}(t),\ \mathcal{F}_{ABx_{2n},Ix_{2n}}(t),\ \mathcal{F}_{STu,Ju}(kt))} \psi(t)dt \geq 0.$$

Letting $n \rightarrow \infty$ and using (2.8) and (2.9), we get

$$\int_0^{\phi(\mathcal{F}_{p,STu}(kt),\ 1,\ 1,\ \mathcal{F}_{STu,p}(kt))} \psi(t)dt \geq 0.$$

Using (a) of Definition 1.21 , we have $\mathcal{F}_{p,STu}(kt) \geq 1$ for all $t > 0$, which gives

$\mathcal{F}_{p,STu}(kt) = 1$. Thus $p = STu$. Therefore, $STu = Ju = p$. Since (ST, J) is weak

compatible, we get

$JSTu = STJu$, that is $STp = Jp$. (2.12)

Step 3.By putting $x = p$, $y = p$ in (2.1) and using (2.11) and (2.12), we obtain

$$\int_0^{\phi(\mathcal{F}_{ABp,STp}(kt),\ \mathcal{F}_{Ip,Jp}(t),\ \mathcal{F}_{ABp,Ip}(t),\ \mathcal{F}_{STp,Jp}(kt))} \psi(t)dt \geq 0,$$

that is,

$$\int_0^{\phi(\mathcal{F}_{ABp,STp}(kt),\ \mathcal{F}_{ABp,STp}(t),\ 1,\ 1)} \psi(t)dt \geq 0.$$

As φ is non-decreasing in first argument, we have

$$\int_0^{\phi(\mathcal{F}_{ABp,STp}(kt),\ \mathcal{F}_{ABp,STp}(t),\ 1,\ 1)} \psi(t)dt \geq 0.$$

Using (b) of Definition 1.21, we have $\mathcal{F}_{ABp,STp}(t) \geq 1$ for all $t > 0$, which gives

$\mathcal{F}_{ABp,STp}(t) = 1$. Thus $ABp = STp$.

Therefore $p = ABp = STp = Ip = Jp$, that is p is a common fixed point of AB, ST, I and J.

Uniqueness. Let q be another common fixed point of AB, ST, I and J.

Then $q = ABq = STq = Iq = Jq$.

By putting $x = p$ and $y = q$ in (2.1) , we get

$$\int_0^{\phi(\mathcal{F}_{ABp,STq}(kt),\ \mathcal{F}_{Ip,Jq}(t),\ \mathcal{F}_{ABp,Ip}(t),\ \mathcal{F}_{STq,Jq}(kt))} \psi(t)dt \geq 0,$$

that is

$$\int_0^{\phi(\mathcal{F}_{p,q}(kt),\ \mathcal{F}_{p,q}(t),\ 1,\ 1)} \psi(t)dt \geq 0.$$

As φ is non-decreasing in first argument, we have

$$\int_0^{\phi(\mathcal{F}_{p,q}(kt),\ \mathcal{F}_{p,q}(t),\ 1,\ 1)} \psi(t)dt \geq 0.$$

Using (a) of Definition 1.21, we have $\mathcal{F}_{p,q}(t) \geq 1$ for all $t > 0$,

which gives $\mathcal{F}_{p,q}(t) = 1$, that is $p = q$.

Therefore, p is the unique common fixed point of the self-maps AB, ST, I and J.

Finally, we need to show that p is also a common fixed point of A, B, S, T, I and J. For this let p be the unique common fixed point of both the pairs (AB, I) and (ST, J).

Then by using commutativity of the pairs (A, B), (A, I) and (B, I), we obtain

Ap = A(ABp) = A(BAp) = AB(Ap), Ap = A(Ip) = I(Ap),

Bp = B(ABp) = B(A(Bp)) = BA(Bp) = AB(Bp), Bp = B(Ip) = I(Bp) ,

which shows that Ap and Bp are common fixed point of (AB, I), yielding thereby

$$Ap = p = Bp = Ip = ABp \qquad (2.13)$$

in the view of uniqueness of the common fixed point of the pair (AB , I).

Similarly using the commutativity of (S, T), (S, J) and (T, J), it can be shown that

$$Sp = Tp = Jp = STp = p . \qquad (2.14)$$

Now we need to show that Ap = Sp and Bp = Tp.

For this put x = p and y = p in (2.1) and using (2.13) and (2.14),we get

$$\int_0^{\phi(\mathcal{F}_{ABp,STp}(kt),\ \mathcal{F}_{Ip,Jp}(t),\ \mathcal{F}_{ABp,Ip}(t),\ \mathcal{F}_{STp,Jp}(kt))} \psi(t)dt \geq 0,$$

that is,

$$\int_0^{\phi(\mathcal{F}_{Ap,Sp}(kt),\ \mathcal{F}_{Ap,Sp}(t),\ \mathcal{F}_{Ap,Ap}(t),\ \mathcal{F}_{Sp,Sp}(kt))} \psi(t)dt \geq 0 .$$

As φ is non-decreasing in first argument , we have

$$\int_0^{\phi(\mathcal{F}_{Ap,Sp}(kt),\ \mathcal{F}_{Ap,Sp}(t),\ 1,\ 1)} \psi(t)dt \geq 0.$$

Using (b) of Definition 1.21 , we obtain

$\mathcal{F}_{Ap,Sp}(t) \geq 1$ for all $t > 0$, which gives $\mathcal{F}_{Ap,Sp}(t) = 1$, that is Ap = Sp .

Similarly it can be shown that Bp = Tp.

Thus p is the unique common fixed point of A, B, S, T, I and J .

This completes the proof.

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