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NORMAL (GRAVIMETRIC) HEIGHTS VERSUS ORTHOMETRIC HEIGHTS

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Abstract: *Since the existing geodetic and levelling networks don't comply with the required accuracy for the national network covering the area of Sudan, a precise and consistent uniform geodetic and leveling control network was established beginning from Halfa town to Kajbar at the northern boarder of Sudan towards Shereik , Sabaloka and Upper Atbara at the Southern -East of Sudan. This leveling line is established for dams construction (Kajbar-Dal, Shereik, Sabaloga and Satit-Atbara Dams) and irrigations project. Based on the geometric height differences and gravimetric observations processing of the control points, normal heights were computed. Consistency of the computed normal heights with the existing ones for the documented existent levelling benchmarks which have been tied was checked. Due to some differences between the computed normal heights and the existing elevations, a new height reference system was defined and realized. This new system is a normal height system, with GRS80 as the reference ellipsoid. Tables presented describe the results obtained and the achieved accuracy. Comparison between normal heights and orthometric heights were shown as the main purpose of this paper.*

Keywords: *Orthometric heights, normal heights, GRS80, geometric height, geopotential.*

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1. INTRODUCTION

The interest of gravimetric observations on a levelling network is that they enable to compute geopotential differences between the gravimetric points. And whereas the geometric height difference between two points may depend on the levelling way, the geopotential difference between these points is unique. So it is preferable to make the adjustment of a levelling network with geopotential differences rather than with the raw geometric height differences. The common concept of altitudes correspond to a well defined physical quantity. The Earth's gravity potential V. indeed, physically horizontal surfaces are surface were the gravity potential is constant. And if an object (or water) is located at a point with geopotential number V_0 , it will aim at falling (or flowing) towards places geopotential is less than V_0.

The geopotential difference between two points A and B is theoretically defined by

$$V_B - V_A = \int_A^B \bar{g}.d\bar{s},$$

Where $d\bar{s}$ is the elementary displacement along the way from A to B and $\bar{g}$ is the gravitational acceleration along this way. In practice, the geopotential difference between two gravimetric and leveled points can obtained by:

$$V_B - V_A = \frac{g_A + g_B}{2} \times \delta h_A^B \qquad (1)$$

Where g_A and g_B are the gravity values on these points and δh_A^B is the geometrical height difference measured by leveling [3]. On each leveled zone in Sudan (Upper Atbara, Shereik, Kajbar and Sabaloka), geopotential differences were thus computed between adjacent gravimetric point using equation (1). As the three leveling networks include loops, these raw geopotential differences had to be adjusted. We then obtained adjust geopotential differences between adjacent gravimetric points of three networks.These adjusted geopotential differences are the most suitable quantities to describe the "physical reality". However, they cannot be used as they are for civil engineering or for a nati onal height reference system. Indeed, they must be related to reference point so that heights can be computed (and not height differences). Moreover, their SI unit is m^2 / s^2 whereas heights are expected to be expressed in meters. For these two reasons, a height reference system had to be defined. The interest of gravimetric observations on a levelling network is that they enable to compute geopotential differences between the gravimetric points. And whereas the geometric height difference between two points may depend on the levelling way, the geopotential difference between these points is unique. So it is preferable to make the adjustment of a levelling network with geopotential differences rather than with the raw geometric height differences [4].

2. ORTHOMETRIC HEIGHTS

Let M_0 be the projection of point *M* on the geoid along the gravity field line which crosses *M.* In the case of orthometric heights, the theoretical value for γ^*_M is the mean value of g along the field line $\overline{M_0M}$

$$\tilde{g}_m = \frac{1}{\overline{M_0M}} \frac{\int g.ds}{\overline{M_0M}} = \frac{C_M}{\overline{M_0M}}.$$

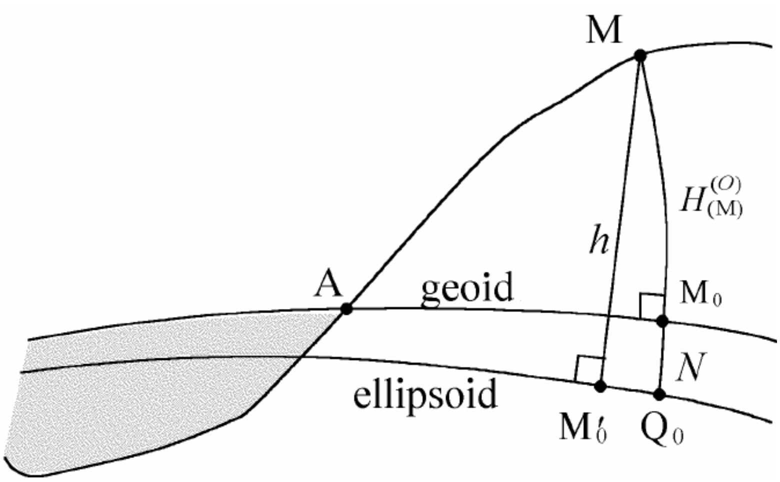

Fig. (1): Difiention of orthometric heights

The orthometric height of point *M* is thus $H^O_M = \frac{C_m}{\tilde{g}_M} = \overline{M_OM}$ (2)

It is the length height of the line of force which links *M* to the geoid. This definition shows that orthometric heights also have a physical and geometrical meaning , even if they are not equivalent to the gravitational potential. However, there is no way to compute an exact orthometric height. Indeed, to determine the mean gravity value $\tilde{g}_M$ along the field line $\overline{M_OM}$, one should know the gravity value everywhere on this line, which is impossible.

In practice, g is supposed to vary linearly along the field line , so that it can be expressed as:

$\tilde{g}_M = g_M - \frac{1}{2} H_M^O \times \left.\frac{\partial g}{\partial H}\right)_{MOY}$, where $\left.\frac{\partial g}{\partial H}\right)_{MOY}$ is the mean gravity gradient along the field line $\overline{M_O M}$. The orthometric height of point *M* can now be computed by:

$$H_M^O = \frac{C_M}{g_M}\left[1 + \left.\frac{\partial g}{\partial H}\right)_{MOY} \times \frac{C_M}{2g^2{}_M}\right] \qquad (3)$$

But once again, there is no way to compute the exact mean gravity gradient $\left.\frac{\partial g}{\partial H}\right)_{MOY}$ unless we dispose of DTM and of the density of the terrain. Usually, this gravity gradient is thus set to a constant: $\left.\frac{\partial g}{\partial H}\right)_{MOY}$ = - 0,848.10^{-6} s^{-2}, the so called (Poincare – Prey gradient) [5]. So even if orthometric heights theoretically have a physical meaning, there is no way to compute them exactly. Approximations have to be done so that computed orthometric heights do not reflect any physical reality anymore.

3. NORMAL HEGHTS

In the case of normal heights, $\gamma^*{}_M$ is not referred to the real gravity field (like for orthometric heights), but to a theoretical gravity field, called "normal gravity field" and defined as follows.

a) The normal gravity field

The normal gravity field is a model of the Earth's gravity field such as:

i) One of this equipotential surfaces is a geodetic ellipsoid (for example GRS80).

ii) The normal potential on this ellipsoid equals the real potentials on the geoid.

iii) This ellipsoid rotates at the same rate as the earth.

iv) This ellipsoid has the same mass as the earth + the atmosphere.

The reference ellipsoid GRS80 can be defined by four parameters:

i) Its half major axis a = 6378137m.

ii) Its dynamic form factor $J_2 = 1.08263 \times 10^{-3}$.

iii) Its rotational rate $\omega = 7.292115 \times 10^{-5}$ rad / s.

iv) The gravitational constant $GM = 3.986005 \times 10^{14}\ m^3 / s^2$.

From these four fundamental constants, other parameters can be derived:

The first excentricity e, the second excentricity e′ and the parameter q_0 which can be obtained by applying the following formulas:

- $q_0 = \frac{1}{2}\left[\left(1+\frac{3}{e'^2}\right)\arctan e' - \frac{3}{e'}\right]$
- $e = \sqrt{3J_2 + \frac{2}{15}\frac{\omega^2 a^3}{GM}\frac{e^3}{q_0}}$
- $e' = \sqrt{\frac{e^2}{1-e^2}}$

- the geometrical flattening f = $1-\sqrt{1-e^2}$
- the half minor axis b = a (1-f).
- $m = \frac{\omega^2 a^2 b}{GM}$
- $q_o' = 3\left(1+\frac{1}{e'^2}\right)\left(1-\frac{\arctan e'}{e'}\right) - 1$
- the normal gravity at the equator $\gamma_E = \frac{GM}{ab}\left(1 - m - \frac{me' q_o'}{6q_0}\right)$
- the normal gravity at the poles $\gamma_P = \frac{GM}{a^2}\left(1 + \frac{me' q_o'}{3q_0}\right)$

At point M_0 at latitude φ on the reference ellipsoid E_0, we can now compute the normal gravity thanks to Somigliana's formula:

$$\gamma_0 = \frac{a\gamma_E \cos^2\varphi + b\gamma_P \sin^2\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} \qquad (4)$$

b) Definision of normal height

Let us define a spheropotential surface as an equipotential surface of the normal gravity field. Now, let Q be the projection of *M* on the spheropotential surface with normal potential V_M , and let Q_0 be the projection of *M* on the) Definition of normal heights reference ellipspod.

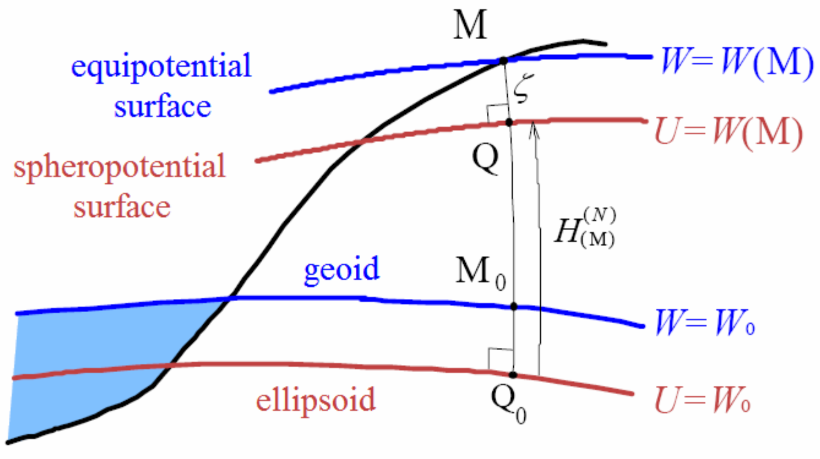

Fig. (2): Definition of normal heights

In the case of normal heights, γ^*_M is the mean normal gravity value $\tilde{\gamma}_M$ along the field line $\overline{Q_0Q}$, so that the normal height of point M is the length of this field line:

$$H_M^N = \frac{C_M}{\tilde{\gamma}_M} \overline{Q_0Q}$$

c) Computation of normal heights

In opposition to $\tilde{g}_M$, the mean normal gravity value $\tilde{\gamma}_M$ is a theoretical quantity and can thus be exactly computed by the following formula:

$$\tilde{\gamma}_M = \gamma_0 \left[1 - \frac{H_M^N}{a} (1 + f + m - f \sin^2 \varphi) + \left(\frac{H_M^N}{a} \right)^2 \right]$$, where φ is the latitude of point *M*.

By replacing $\tilde{\gamma}_M$ by its expression in the definition of a normal height, one can obtain an exact formula for the computation of normal heights:

$$H_M^N = \frac{C_M}{\gamma_0} \left(1 + (1 + f + m - 2f \sin^2 \varphi) \frac{C_M}{a\gamma_0} + \left(\frac{C_M}{a\gamma_0} \right)^2 \right) \quad (5)$$

So, normal height do not refer to a physical reality since they represent the length of a theoretical (normal) line of force. But their first advantage over orthometric heights is that they can be computed exactly.

4. COMPUTATION OF NORMAL AND ORTHOMETRIC HEIGHTS

The geopotential differences had been computed and adjusted previously between adjacent gravimetric points of Kajbar, Shereik, Sabaloka and Upper Atbara levelling networks. The next step was to define a height reference system for each of these four zones and to compute the gravimetric points heights. In fact to satisfy Sudan national demand, two height reference systems were defined for each zone, one with normal heights and the other with orthometric heights. Now let us examine how these systems were defined and how the gravimetric points heights were computed.

4.1 NORMAL HEIGHTS SYSTEMS

First of all, in each of the four zones, a point *A* (for which an old (mean sea level height from Alexandria) height was available) was chosen as the reference point for the new normal height system. Its normal height in the new system was set equal to its height in the old system:

$$H^N_{IGN}(A) = H^0{}_{ALEX}(A)$$

This reference point was KD11 in Kajbar (with 221.8596m of the same normal and orthometeric height), S1 in Shereik (with 356.3451m of the same normal and orthometeric height), RM06 in Sabaloka (with 402.0619m of the same normal and orthometeric height)and RM07 in Upper Atbara (with 491.8460 m of the same normal and orthometeric height). The geopotential number of this point in the new system was computed using the following formula:

$$C_A = H^N_{IGN}(A)\left[1 + (1 + f + m - 2f\sin^2\varphi)\frac{H^N_{IGN}(A)}{a} + \left(\frac{H^N_{IGN}(A)}{a}\right)^2\right] (inverse\,of\ Equation\ (5))$$

For the gravimetric points, geodetic coordinates in the ITRF2005 reference frame were used. Then, using the adjusted geopotential differences, a geopotential number C_M Was assigned to each gravimetric point *M* of the networks. Finally, these geopotential numbers were transformed into normal heights using equation (5).

4.2 ORTHOMETRIC HEIGHTS SYSTEMS

For the orthometric heights systems, the same reference points were used. This time, their orthometric heights in the new system were set equal to their heights in the old system:

$$H^O_{IGN}(A) = H^0_{ALEX}(A)$$

The geopotential numbers of these points in the new system were computed using the following formula:

$C_A = H^{O}_{IGN}(A)\left[g_A - \frac{1}{2}H^{O}_{IGN}(A)\left(\frac{\partial g}{\partial H}\right)_{MOY}\right]$ (inverse of equation (3) and the Poincare Prey gradient).

Then, using the adjusted geopotential differences, a geopotential number C_M was assigned to each gravimetric point *M* of the networks. Finally, these geopotential numbers were transformed into orthometric heights using equation (3).

5. RESULTS AND ANALYSIS

Tables 2, 3, 4 and 5 bellow show the points geodetic coordinates with different kinds of heights (ellipsoid, normal and orthometric heights with the difference between normal and orthometric height of each point). Each table represent one of the four zones (Kajbar-Dal, Shereik, Sabaloka, and Upper Atbara) over the area of Sudan. Table (1), sums up differences between the computed normal and orthometric heights. It contains the biggest normal – orthometric differences for each leveling zone. These differences are expressed in millimeters.

Table (1): Max. normal height – orthometric height differences for each leveling zone

Levelling zone	Kajbar-Dal	Shereik	Sabaloka	Upper Atbara
Maximum differenc	3.6 mm	5.4 mm	7.5 mm	12.4 mm
Comments	There is no meaning to the positive or negative sign here (it is the matter of difference only).			

The overall maximal difference is 12.4 mm, showing that there is no such an important difference between the two kinds of heights. Choosing one or another should not affect engineering works. But it will have a certain influence at the scale of a national leveling network.

6. CONCLUSIONS

Theoretically, orthometric heights have a physical meaning (length of a line of force of the real gravity field). But in practice, they can be computed only with approximate formulas, so that they do not reflect any physical meaning any more. (In reality, g does not vary linearly along the line of force and the gravity gradient is neither constant nor equal to the Poincare-Prey gradient). Normal heights have no physical meaning since they represent the length of a line of force of the normal gravity field. But their first advantage over orthometric heights is that they can be computed exactly. Eventually, it may be preferable to use normal heights exactly than orthometric heights computed with approximations.

7. ACKNOWLEGMENT

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Table (2): Orthometric and gravimetric normal height of Kajbar-Dal

	Latitude North	Longitude East	Ellipsoid height (m)	Elevation (m)		Differences (m)
Point No.	Deg. Min. Sec.	Deg. Min. Sec.		Normal height	Orthometric height	(Normal – Orthometric)
K1	21 11 19.05410	30 40 30.43410	197.525	188.1627	188.1653	-0.0026
K2	21 02 42.55259	30 36 42.01255	199.592	190.3771	190.3807	-0.0036
K3	20 49 26.18267	30 32 28.12264	210.514	201.3646	201.3666	-0.0020
K4	20 48 08.55655	30 19 27.63576	215.222	205.8525	205.8533	-0.0008
K5	20 43 40.20178	30 21 25.47696	220.083	211.4610	211.4622	-0.0012
K6	20 20 13.63744	30 34 19.91158	241.306	232.1627	232.1621	0.006
K7	20 13 13.09482	30 33 15.16197	229.501	220.3782	220.3774	0.0008
K8	20 04 43.03022	30 35 26.19307	228.701	219.8263	219.8267	-0.0004
K9	19 57 04.24771	30 18 56.19263	231.143	222.1939	222.1956	-0.0017
K10	19 42 13.59736	30 23 54.68235	230.047	221.4389	221.4396	-0.0007
The maximum (normal height – orthometric height) difference				3.6 millimetres		

Table (3): Orthometric and gravimetric normal height of Shereik

	Latitude North	Longitude East	Ellipsoid height (m)	Elevation (m)		Differences (m)
Point No.	Deg. Min. Sec.	Deg. Min. Sec.		Normal height	Orthometric height	(Normal – Orthometric)
S1	18 55 13.43177	33 31 05.88341	346.152	339.7508	339.7510	-0.0002
S2	19 06 50.48011	33 35 31.55291	336.733	330.1966	330.1983	-0.0017
S3	19 19 56.01406	33 22 16.42551	333.336	326.3343	326.3326	0.0017
S4	19 29 43.20861	33 08 52.39356	325.509	318.1891	318.1914	-0.0023
S5	18 42 59.63383	33 42 30.31540	358.899	353.0427	353.0442	-0.0015
S6	18 29 29.90828	33 42 30.46582	357.114	351.4044	351.4055	-0.0011
S7	18 17 54.37226	33 55 49.79934	354.174	349.0708	349.0754	-0.0046
S8	17 59 10.25947	33 57 31.55094	351.304	346.4869	346.4917	-0.0048
S9	17 49 40.60139	33 59 56.64325	352.924	348.2418	348.2457	-0.0039
S10	17 44 11.71484	33 59 07.39040	353.454	348.9183	348.9237	-0.0054
The maximum (normal height – orthometric height) difference				5.4 millimetres		

Table (4): Orthometric and gravimetric normal height of Sabaloka

	Latitude North	Longitude East	Ellipsoid height (m)	Elevation (m)		Differences (m)
	Deg. Min. Sec.	Deg. Min. Sec.		Normal height	Orthometric height	(Normal – Orthometric)
B1	16 09 19.37499	32 33 01.89511	387.074	383.8159	383.8133	0.0026
B2	15 53 28.63554	32 31 37.49697	386.480	383.3823	383.3777	0.0046
B3	15 28 55.98397	32 24 20.08847	388.595	385.9885	385.9848	0.0037
B4	15 23 05.13550	32 46 27.04893	387.380	385.3510	385.3463	0.0047
B5	15 17 56.72386	32 26 49.57072	385.309	382.9513	382.9438	0.0075
B6	16 32 27.58459	32 51 12.10306	369.924	366.4050	366.4050	0.0000
B7	16 32 27.42167	33 04 54.61049	371.498	367.9843	367.9804	0.0039
B8	16 20 28.76946	32 44 22.00272	386.375	383.0267	383.0263	0.0004
B9	15 59 27.85410	32 35 16.07217	383.813	380.6408	380.6367	0.0041
B10	15 35 50.02975	32 36 23.70486	386.975	384.3374	384.3314	0.0060
The maximum (normal height – orthometric height) difference				7.5 millimetres		

Table(5): Orthometric and gravimetric normal height of Upper-Atbara

	Latitude North	Longitude East	Ellipsoid height (m)	Elevation (m)		Differences (m)
Point No.	Deg. Min. Sec.	Deg. Min. Sec.		Normal height	Orthometric height	(Normal – Orthometric)
A1	14 16 20.75905	36 30 38.29566	572.562	574.0622	574.0722	-0.0100
A2	13 42 54.63437	36 13 10.81828	565.960	568.3148	568.3232	-0.0084
A3	14 02 21.83972	35 56 45.96610	529.873	531.9654	531.9778	-0.0124
A4	15 18 11.47605	36 16 16.96283	504.882	504.9327	504.9358	-0.0031
A5	15 30 48.22169	36 02 09.78810	461.645	461.2274	461.2171	0.0103
A6	15 38 46.33500	36 10 24.35259	464.768	464.4932	464.4971	-0.0039
A7	15 32 43.74890	36 16 04.46566	486.783	486.4434	486.4350	0.0084
A8	14 22 15.58766	35 53 25.23242	511.180	512.5221	512.5243	-0.0022
A9	14 45 12.60291	35 57 56.46784	492.807	493.5382	493.5397	-0.0015
A10	15 01 48.15647	35 56 00.42438	464.436	464.8682	464.8758	-0.0076
The maximum (normal height – orthometric height) difference				12.4 millimetres		