APPLICABILITY PROPERTIES OF EIGENVALUES AND EIGENVECTORS IN CLASSICAL MATRICES – RELATEVELY ALLOW IN FUZZY MATRICES

Prof. A. N. Mohamad, Department of Mathematics and Modeling, College of Natural & Computational Sciences, Debre Berhan University, Ethiopia

Gebrehiwot, Department of mathematics, College of Natural & Computational Sciences, Debre Berhan University, Ethiopia

Beharu Getachew, M.Sc. (Algebra) Student, Department of Mathematics, College of Natural & Computational Sciences, Debre Berhan University, Ethiopia

Abstract: *The Eigen value problem and Eigen vector Problems (EVP) are considered as theoretical way and have wide – range applications. Especially EVP is crucial in solving system of differential equations, Analyzing population growth Models etc. The problem of finding eigenvalue and eigenvector arise in wide variety of practical applications. The aim of this article to determine Fuzzy Eigen Values and fuzzy Eigen Vectors of fuzzy matrices: we will explain here some properties of classical matrix eigenvalues and eigenvectors are satisfied in Fuzzy matrix and also prove those properties in this article.*

Keywords: *Fuzzy numbers, fuzzy eigenvalues, fuzzy eigenvectors, triangular fuzzy numbers, and Fuzzy intervals*

1. INTRODUCTION

It is well known, that matrices play major role in various areas such as mathematics, physics, statistics, engineering, social sciences and many others. Several works on classical matrices are available in different journals even in books also. But in daily life situations, the problem in economics, engineering, environment, social science, medical science etc. They do not always involve crisp data. Consequently, we cannot successfully use traditional classical matrices because of various types of uncertainties present in daily life problems. Now a day's probability, fuzzy sets, intuitionistic fuzzy sets, vague sets, rough sets are used as mathematical tools for dealing uncertainties. Fuzzy matrices arise in many applications, one of which is properties of eigenvalues and eigenvectors applicability in Fuzzy Matrices. Its applications in pattern classification in handling fuzzy knowledge systems

Reality is more or less uncertain, vague and ambiguous. Fuzzy logics are powerful mathematical tools for modeling, uncertain systems in Industry, Nature and Humanity and Facilitators for common – sense reasoning in decision making in the absence of complete and precise information. Their role is significant when applied to complex phenomena not easily described by traditional mathematical methods.

Fuzzy sets and fuzzy logic founded in the 1965 by Lotfi A. Zadeh [10], Bellman 1965[1] can be viewed as a broad conceptual framework enclosing the classical sets and logic. The classical sets and mathematical logic divide the world of "Yes and No", "White and Black" "true and false'. Fuzzy sets and fuzzy logic deal with objects that are "Matter – of – degree" will all possible grades of truth between yes and no, and the shares of grey between white and black.

The basic concept of the fuzzy matrix theory is very simple and can be applied to social and natural situations. A branch of fuzzy matrix theory uses algorithms and algebra to analyze data. It is used by social scientists to analyze interaction between actors and can be used to complement analysis carried out using game theory or other analytical tools.The problem of finding Eigenvalues arises on a wide variety of practical applications. It arises in almost all branches of sciences and engineering.

Many important characteristics of physics and engineering systems, such as stability can be determined just by knowing the nature and location of Eigenvalues. The aim of this paper to determine Fuzzy Eigenvalues and Fuzzy Eigenvectors of a Fuzzy matrix denoted by$\widetilde{A}$, in the whole article: The system of linear equations$\widetilde{A}\,\widetilde{X} = \widetilde{b}$, where the elements$\widetilde{a}_{ij}$, of the matrix$\widetilde{A}$and the elements$\widetilde{b}_i$, of the vector$\widetilde{b}$, they are called **Fuzzy Numbers.** A solution method for finding Fuzzy Eigenvalues and Eigenvectors of Fuzzy matrix$\widetilde{A}$ with fuzzy idempotent has not been given yet. We have to find Fuzzy Eigenvalues and Fuzzy Eigenvectors of a Fuzzy matrix$\widetilde{A}$ by transforming the system$\widetilde{A}\,\widetilde{X} = \widetilde{\lambda}\,\widetilde{b}$

2. PREVIOUS STUDY

In real world, the parameters and variables in fuzzy matrix may be almost uncertain data. Hence, fuzzy matrix has been discussed recently. Lotfi Zadeh A. (1965)[10] introduced fuzzy sets, information and control, Lewis F. C.[12] introduced Cayley – Hamiltonian Theorem and Fader's Model. Buslowicz M.(1984)[2] discussed unversion of characteristic matrix of the time delay systems of neural type. Kaczorek T. (1988)[7] discussed vectors and matrices in Automation and

Electro techniques. Chang F. R and Chan C. N (1992) [4]introduced the generalized Cayley – Hamiltonian Theorem for standard pencils, system and control. Kaczorek T. (1994)[8] discussed extensions of the Cayley – Hamiltonian theorem for 2D continuous discrete linear system. Again Kaczorek T. (1995)[9]introduced an extension of the Cayley – Hamiltonian theorem for non square block matrices and computations of the left and right inverse of matrices.

Zimmermann H. J. (1996)[13]introduced fuzzy set theory and its application. Kaczorek T. (1994)[8]explained the extension of the Cayley – Hamiltonian theorem for a standard pair of block matrices. Buslowicz M and Kaczorek T. (2004)[3] introduced reachability and minimum energy control of positive linear discrete time systems with delay: Michael Hans (2005)[12]introduced applied fuzzy arithmetic: Dehaghan M andHashemi B. (2006)[6]introduced solution of the fully fuzzy linear systems using the decomposition produce**.** Das S. and Charaverly S. (2011)[5]introduced fuzzy system of linear equations and its application in circuit analysis

Several authors presented a number of results on fuzzy matrices and introduced the concept of fuzzy matrices and studied the canonical form of a transitive matrix. They studied the canonical form of an idempotent matrix,presented the canonical form a strongly transitive matrix. And also studied fuzzy circulates matrices.

2.1 Binary Set Operations

Intersection : $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$

Union : $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$

Complement : $\mu_{A^c}(x) \quad = 1 - \mu_A(x)$

"$\wedge$" : $Minimum$

"$\vee$" : $Maximum$

2.2 Unit Interval

A fuzzy matrix is a matrix which has its elements in$[0, 1]$ closed interval called fuzzy unit interval

2.3 Fuzzy Matrix

Consider the matrix $A = \left(a_{ij}\right)_{m\times n}, where\ a_{ij} \in [0,1]\ for\ 1 \le i \le m\ and\ 1 \le j \le n,$ and then A, it is called fuzzy rectangular matrix, and if $m = n$, and then A, it is called fuzzy square matrix.

2.4 Reflexive

Let$A$, it is a square matrix of order$n$, and then$A$, it is said to be reflexive fuzzy matrix if and only if$A \geq I_n$ (WhereI_n, it is an identity matrix of order$n$), that is

If and only if "All diagonal elements in fuzzy reflexive matrix$A$, they are unity otherwise

If and only if "$(a_{ij}) = 1 \; \forall i$ (Where(a_{ij}), it is the elements of$A$

2.5 Symmetric

Let$A$, it is a square matrix of order$n$, and then$A$, it is said to be symmetric fuzzy matrix if and only if$A^T = A$, that is

If and only if "The square fuzzy matrix$A$, remains unaltered by interchanging its rows and columns, otherwise

If and only if "$(a_{ij}) = (a_{ji}) \forall i, j \in \{1, 2, \dots, n\}$

2.6 Transitive

Let A, it is a square matrix of order$n$, and then$A$, it is said to be transitive fuzzy matrix if and only if$A^2 \leq A$, that is

If and only if "The square fuzzy matrix$A$, multiplied by itself give the elements less than or equal to the corresponding elements of the square fuzzy matrix, in other words

If and only if "$a_{ik} a_{kJ} = a_{ij} \forall k \in \{1, 2, \dots, n\}$

2.7 Fuzzy Diagonal Matrix

A fuzzy square matrix A, it is a square matrix of order$n$, it is said to be fuzzy diagonal matrix if

$a_{ij} = \begin{cases} 0, & if \; i \neq j \\ a_{ij} \in [0, 1], & for 1 \leq i, j \leq n \end{cases}$, for example, the order 3 fuzzy diagonal matrix is

$$\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}$$

2.8 Fuzzy Scalar Matrix

A fuzzy square matrix A, it is a square matrix of order$n$, it is said to be fuzzy scalar matrix if all its diagonal entries are equal that is, if

$a_{ij} = \begin{cases} 0, & if \; i \neq j \\ a_{ij} = \alpha, & if \; i = j, \alpha \in [0, 1] and \; 1 \leq i, j \leq n \end{cases}$, for example the order 3 fuzzy scalar matrix is

$$\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

2.9 Transpose of Fuzzy Matrix

Consider the matrix $A = (a_{ij})_{m\times n}$, it is any fuzzy matrix and then the transpose of A, it is denoted byA^T, it is a $n \times m$ fuzzy matrix obtained from A, by interchanging its rows and columns

2.10 Construct Fuzzy Square Matrix

The square fuzzy matrix A, through the adjoint matrix $adj\ (A)$, we shall construct a transitive Fuzzy matrix $A\big(adj\ (A)\big)$, that is

$A\big(adj\ (A)\big) \geq |A|I\ \ and\ \big(adj\ (A)\big)A \geq |A|I\ where\ |A|$, denotes the determinant of square fuzzy matrix $A\ and\ adj\ (A)$, denotes the adjoint matrix of square fuzzy matrix A

2.11 Characteristic Function or Membership Function

If $x \in A$, the corresponding function $\forall x \in E: \mu_A(x) \in [0,1]$, the function$\mu_A(x)$, it is called the characteristic function or membership function, note that the subset at function $E\ on\ [0,1]$

2.12 Empty Subset

The empty subset $\emptyset$ as $\forall x \in E: \mu_\emptyset(x) = 0$

2.13 Power set

The set E, itself $\forall x \in E: \mu_E(x) = 1$, then the set of all subsets is called the power set and denoted by $P(E)\ \ or\ [0,1]^E$

2.14 Membership Value matrices

Letf_{mn}, denotes the set of all $m \times n$ matrices over f , and if $m = n$, we havef_n, and then the elements off_{mn}, they are called membership valued matrices or binary fuzzy relation matrices, simply fuzzy matrices or Boolean matrices over Boolean algebra$[0,1]$

2.15 Addition of Fuzzy matrices

Let $A = (a_{ij}) \in f_{mn}\ and\ B = (b_{ij}) \in f_{mn}\ then\ the\ matrix\ A + B = \sup\{a_{ij}, b_{ij}\} \in f_{mn}$, it is called the sum of fuzzy matrices $A\ and\ B$, for example if

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.8 & 0.2 & 0.3 \\ 0 & 0.6 & 0.1 \end{bmatrix}; B = \begin{bmatrix} 0.2 & 0.4 & 0.6 \\ 0.5 & 0.3 & 0.3 \\ 0.7 & 0.8 & 0 \end{bmatrix}\ then\ A + B = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.8 & 0.3 & 0.3 \\ 0.7 & 0.8 & 0.1 \end{bmatrix}$$

2.16 Basic requirements of fuzzy Number u in parametric for

Fuzzy Number u in parametric form is a pair$(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r)$ for $0 \leq r \leq 1$, which satisfies the following requirements.

a. $\underline{u}(r)$, it is a bounded non decreasing left continuous function in$(0,1]$ and right continuous at 0

b. $\overline{u}(r)$, it is a bounded non decreasing right continuous function in$[0,1)$ and left continuous at 1

c. $\underline{u}(r) \leq \overline{u}(r)$ for $0 \leq r \leq 1$

2.17 Triangular Fuzzy number

The triangular fuzzy number $u = (x_0, \sigma, \beta)$ (Fuzzy number with a defuzzifierx_0, left fuzziness $\sigma > 0$ and right fuzziness $\beta > 00$ it is a fuzzy set where the membership function is as follows:

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma), \; for\ x_0 - \sigma \leq x \leq x_0 \\ x\frac{1}{\beta}(x_0 - x + \beta), \; for\ x_0 \leq x \leq x_0 + \beta \\ 0, \; Otherwise \end{cases}$$

We can easily verify that

$[u]^r = [x_0 - (1-r)\sigma, x_0 + (1-r)\beta] = [x_0 - \alpha_1(r), x_0 + \alpha_2(r)], \forall\, r \in [0,1]$, where

$\alpha_1(r) = (1-r)\sigma$ and $\alpha_2(r) = (1-r)\beta$

2.18 Support of Fuzzy number

Let$u$, it is a fuzzy number and then the support of$u$, it is defined by$\sup(u) = \overline{\{x|u(x) > 0\}}$

where$\overline{\{x|u(x) > 0\}}$, represents the closure of$\{x|u(x) > 0\}$

2.19 Addition, Subtraction and multiplication of two fuzzy numbers

Addition of two fuzzy numbers

Suppose that$\tilde{u}(r) = [\underline{u}(r), \overline{u}(r)]$, and $\tilde{v}(r) = [\underline{v}(r), \overline{v}(r)]$, and then

$$(\tilde{u} + \tilde{v})(r) = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]\ for\ 0 \leq r \leq 1$$

Subtraction of two fuzzy numbers

Suppose that$\tilde{u}(r) = [\underline{u}(r), \overline{u}(r)]$, and $\tilde{v}(r) = [\underline{v}(r), \overline{v}(r)]$, and then

$$(\tilde{u} - \tilde{v})(r) = [\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)]\ for\ 0 \leq r \leq 1$$

Multiplication of two fuzzy numbers

Suppose that$\tilde{u}(r) = [\underline{u}(r), \overline{u}(r)], and\ \tilde{v}(r) = [\underline{v}(r), \overline{v}(r)]$, and then

$$(\tilde{u}\tilde{v})(r) = \begin{bmatrix} \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), & \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\} \\ \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), & \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\} \end{bmatrix}, for\ 0 \le r \le 1$$

2.20 Eigenvalue and Eigenvector of Fuzzy number

Let$\tilde{A} = (\tilde{a}_{ij})$, it is a fuzzy matrix and a fuzzy number$\tilde{\lambda}$and a fuzzy vector$\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$, which is given by$\tilde{x}_i(r) = [\underline{x_i}(r), \overline{x_i}(r)]\ for\ 1 \le i \le n\ and\ 0 \le r \le 1$, and then they are called eigenvalue and eigenvector of fuzzy matrix$\tilde{A}$, respectively if

$$\underline{\sum_{i=1}^{n} \tilde{a}_{ij}\, \tilde{x}_j} = \sum_{i=1}^{n} \underline{\tilde{a}_{ij}\, \tilde{x}_j} = \underline{\tilde{\lambda}\tilde{x}_i};\ \overline{\sum_{i=1}^{n} \tilde{a}_{ij}\, \tilde{x}_j} = \sum_{i=1}^{n} \overline{\tilde{a}_{ij}\, \tilde{x}_j} = \overline{\tilde{\lambda}\tilde{x}_i}$$

2.21 Fuzzy Set

A fuzzy set A , it can be defined as the set of ordered pairs such that

$A = \{(x, \mu_A(x)) | x \in X, \mu_A(x) \in [0,1]\}\ where\ \mu_A(x)$, it is called the membership function or grade of membership of x

2.22 Convex normalized fuzzy set

A fuzzy number μ, it is a convex normalized fuzzy set of the crisp set such that for only one $x \in X\ amd\ \mu_A(x) = 1\ and\ \mu_A(x)$, it is piecewise continuous

2.23 Normalized

A fuzzy set is called normalized when at least one its elements attains the maximum possible membership grade

2.24 Convex Fuzzy set

A convex fuzzy set is described by a membership function whose membership values are strictly monotonically increasing or whose membership values are strictly monotonically decreasing or whose membership values are monotonically increasing and then strictly monotonically decreasing with increasing values for elements in the universe

2.25 Types of Fuzzy numbers in the interval

Here we discussed only two types of fuzzy numbers in the interval namely

a. Triangular fuzzy number in interval
b. Trapezoidal fuzzy number in interval

2.26 Triangular Fuzzy number in interval

We can show graphically as follows

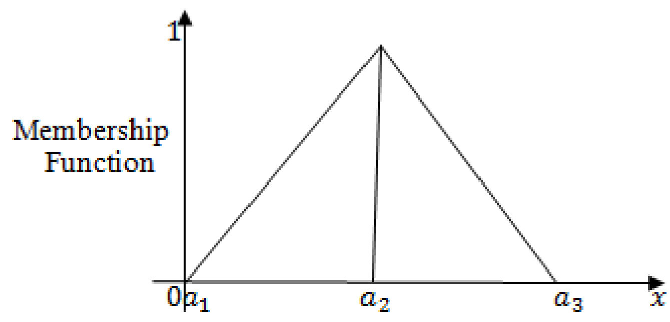

Figure – 1: Triangular fuzzy number in interval

The triangular fuzzy number (TFN) as shown in figure – 1, it is a special type of fuzzy number and its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & for\ x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & for\ x \in [a_1, a_2] \\ \frac{a_3-x}{a_3-a_2}, & for x \in [a_2, a_3] \\ 0, & for\ x \geq a_3 \end{cases}$$

2.27 Trapezoidal Fuzzy number in interval

We can show graphically as follows

Figure – 2: Trapezoidal fuzzy number in interval

The trapezoidal fuzzy number (TRFN) as shown in figure – 2, it is a special type of fuzzy number and its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & for\ x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & for\ x \in [a_1, a_2] \\ 1, & for x \in [a_2, a_3] \\ \frac{a_4-x}{a_4-a_3}, & for x \in [a_3, a_4] \\ 0, & for\ x \geq a_4 \end{cases}$$

In this article we will discuss properties of eigenvalues and eigenvectors of fuzzy matrices

3.1. Needed fundamental Definitions and preliminaries

3.1.1. Definition for Fully eigenvectors and eigenvalues

Let A_F, denoted as Fuzzy matrix A and also $A = [a_{ij}]$, it is a square matrix of order n.

If there exists a non – zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, such that $A_F X = \lambda X where X$, it is called

Eigenvector of $A_F and \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$, it is the corresponding eigenvalues

3.1.2. Note

1. Corresponding to n distinct fuzzy eigenvalues, we get n independent eigenvectors.
2. If two or more fuzzy eigenvalues are equal, then it may or may not possible to get linearly independent eigenvectors corresponding to the repeated fuzzy eigenvalues.
3. If x_i, it is the solution for an eigenvalue λ_i, and then it follows from $(A_F - \lambda I)X = 0$, that cx_i, it is also a solution, where c, it is an arbitrary constant. Thus, the eigenvector corresponding to a fuzzy eigenvalue is not unique but may be one of the vectors cX
4. Algebraic multiplicity of an eigenvalue λ it is the order of the fuzzy eigenvalue as root of the characteristic polynomial $(A_F - \lambda I)X = 0$

   That is, if λ it is the double root then algebraic multiplicity is 2
5. Geometric multiplicity of λ it is the number of linearly independent eigenvectors corresponding to λ

3.1.3. Working Rule to find Eigenvalues and eigenvectors of Fuzzy Matrix

Step – 1:

Find the characteristic equation $|A_F - \lambda I| = 0$

Step – 2:

Solving the characteristic equation, we get characteristic roots. They are called fuzzy eigenvalues

Step – 3:

To find eigenvectors solve $(A_F - \lambda I)X = 0$, for the different values of λ

3.1.4. Non – Symmetric Fuzzy Matrix

If a fuzzy matrix A_F, it is non – symmetric then $A_F \neq A_F^T$

Note:

1. In a non – symmetric fuzzy matrix, the fuzzy eigenvalues are non – repeated then we get linearly independent set of eigenvectors.
2. In a non – symmetric fuzzy matrix the fuzzy eigenvalues are repeated then we may or may not be possible to get linearly independent eigenvectors, and then Diagonalization is possible through similarity transformation

3.1.5. Symmetric Fuzzy Matrix

If a fuzzy matrix A_F, it is symmetric then $A_F = A_F^T$

Note:

1. In a symmetric fuzzy matrix, the fuzzy eigenvalues are non – repeated then we get linearly independent set of eigenvectors and pair wise orthogonal sets of eigenvectors
2. In a symmetric fuzzy matrix the fuzzy eigenvalues are repeated then we may or may not be possible to get linearly independent eigenvectors, and pair wise orthogonal sets of eigenvectors, the Diagonalization is possible through orthogonal transformation

3.2. Some Properties of Eigenvalues and Eigenvectors of Fuzzy Matrix

3.2.1. Property – 1:

a. The sum of the Fuzzy Eigenvalues of a fuzzy matrix is the sum of the elements of the principle (main) diagonal of the Fuzzy matrix (or) the sum of the fuzzy eigenvalues of a fuzzy matrix is equal to the trace of the fuzzy matrix.

b. Product of the Fuzzy Eigenvalues of the fuzzy matrix is equal to the determinant of the fuzzy matrix.

Proof:

$$\text{Let}(A_F - \lambda I)X = 0 \ \ where A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}; \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} and X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (1)$$

And then we get

$$\begin{Bmatrix} (a_{11} - \lambda_1)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda_2)x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ a_{n1}x_1 + a_{12}x_2 + \cdots + (a_{nn} - \lambda_n)x_n = 0 \end{Bmatrix} \quad (2)$$

Since A_F, it is the fuzzy square matrix of order n, and then the characteristic polynomial of $A_F = |A_F - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} (a_{11}-\lambda_1) & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22}-\lambda_2) & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-\lambda_n) \end{vmatrix} = 0 \quad (3)$$

Then we have

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \cdots + S_{n-1}\lambda + (-1)^n S_n = 0 \quad (4)$$

$$Where$$

S_1 =Sum of the diagonal elements of A_F

… …

S_n = Determinant of A_F

Since the roots of the characteristic equation are called fuzzy eigenvalues of the given fuzzy matrix solving (1), we get n roots

Let us take the n roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$, that is they are fuzzy eigenvalues of A_F

Since we have from (4)

$$\left\{ \begin{array}{c} \lambda^n - (sum of the roots)\lambda^{n-1} \\ +(Sum of the product of roots take two at a time)\lambda^{n-2} \\ + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . + \\ (-1)^n(Product of roots) \end{array} \right\} = 0 \quad (5)$$

$\Rightarrow$Sum of roots $= S_1$ and by equations (4) and (5) we have

$$S_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n \Rightarrow \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$$

$\Rightarrow$Sum of fuzzy eigenvalues equal to sum of main diagonal of fuzzy matrix A_F

Hence complete the proof of part (a)

And also we have

S_n =Product of roots$= \lambda_1 * \lambda_2 * \ldots * \lambda_n \Rightarrow \lambda_1 * \lambda_2 * \ldots * \lambda_n = |A_F|$

$\Rightarrow$Product of fuzzy eigenvalues equal to determinant of fuzzy matrix A_F

Hence complete the proof of (b)

3.2.2. Property – 2:

A fuzzy square matrix A_F and its transpose A_F^T, they have the same fuzzy eigenvalues or characteristic values.

Proof:

Let A_F, it is a fuzzy square matrix of order n

Since the characteristic polynomial of A_F and A_F^T, they are

$$|A_F - \lambda I| = 0 \qquad (6)$$

$$|A_F^T - \lambda I| = 0 \qquad (7)$$

Note that in the classical matrix the determinant value is unaltered by interchanging of rows and columns. The same rule is applicable in fuzzy matrix also that is

$$|A_F| = |A_F^T|$$

Therefore equations (6) and (7) are identical and we will get the eigenvalues of A_F and A_F^T they are same

Hence complete the proof

3.2.3. Property – 3:

The characteristic roots of a triangular fuzzy matrix are just the diagonal elements of the fuzzy matrix. That is the eigenvalues of a triangular fuzzy matrix are just the diagonal elements of the fuzzy matrix.

Proof:

Let us take $A_F = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

The characteristic polynomial of A_F is $|A_F - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda_1 & 0 & \cdots & 0 \\ a_{21} & a_{22} - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda_n \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda_1)(a_{22} - \lambda_2) \ldots (a_{nn} - \lambda_n) = 0$$

$$\Rightarrow (a_{11} - \lambda_1) = 0; (a_{22} - \lambda_2) = 0; \ldots (a_{nn} - \lambda_n) = 0$$

$$\Rightarrow a_{11} = \lambda_1; a_{22} = \lambda_2; \ldots, a_{nn} = \lambda_n$$

That is eigenvalues or characteristic roots of triangular square matrix of order n they are the diagonal elements of A_F

Hence complete the proof

3.2.4. Property – 4:

If $\lambda and \lambda \neq 0$, it is the eigenvalue A_F, and then $\frac{1}{\lambda}$, it is the eigenvalue A_F^{-1} (Inverse of A_F)

Proof:

If X, it is the fuzzy eigenvector corresponding to λ, and then we have

$$A_F X = \lambda X \quad (8)$$

Equation (1): Pre multiplying both sides by A_F^{-1}, we get

$$A_F^{-1} A_F X = A_F^{-1} \lambda X$$

$$\Longrightarrow IX = A_F^{-1} \lambda X \Longrightarrow X = A_F^{-1} \lambda X \quad (9)$$

Divide (9) by λ, since λ, it is a constant, we get

$$\frac{1}{\lambda} X = A_F^{-1} X \quad (10)$$

Equation (9) is same form of (8) so that we have $\frac{1}{\lambda}$ fuzzy eigenvalue of A_F^{-1}

Hence complete the proof

3.2.5. Property – 5

If $\lambda and \lambda \neq 0$, it is the eigenvalue of orthogonal fuzzy matrix A_F, and then $\frac{1}{\lambda}$, it is also the eigenvalue of A_F

Proof:

By the definition of orthogonal fuzzy matrix we have

$A_F A_F^T = A_F^T A_F = I$ (Identity matrix of same order of A_F)

$$\Longrightarrow For A_F A_F^T = I \Longrightarrow A_F^T = \frac{1}{A_F} = A_F^{-1} \Longrightarrow A_F^T = A_F^{-1}$$

Let A_F it is the orthogonal fuzzy matrix. Then A_F^T it is also an orthogonal fuzzy matrix

$\Longrightarrow A_F^{-1}$, it is an orthogonal matrix

Therefore, if $\lambda and \lambda \neq 0$, it is the eigenvalue of orthogonal fuzzy matrix A_F, and then $\frac{1}{\lambda}$, it is the eigenvalue of A_F^{-1} (By property – 4)

But $A_F^T = A_F^{-1}$

Since $A_F and A_F^T$, they have same fuzzy eigenvalues (By property – 2)

$\Longrightarrow$ If $\lambda and \lambda \neq 0$, it is the eigenvalue of orthogonal fuzzy matrix A_F, and then $\frac{1}{\lambda}$, it is also the eigenvalue of A_F

Hence complete the proof

3.2.6. Property - 6

If$\lambda_1, \lambda_2, \dots, \lambda_n$, they are the fuzzy eigenvalues of a fuzzy matrixA_F,of order$n$ and thenA_F^m, it has fuzzy eigenvalues$\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$, where$m$, it is a positive integer

Proof:

Let$\lambda_i fori = 1, \dots, n$, it is the fuzzy eigenvalue of fuzzy matrixA_F, and the corresponding eigenvector isX_i

Then we have

$$A_F X_i = \lambda_i X_i \qquad (11)$$

Since$A_F^2 X_i = A_F(A_{FX_i}) = A_F(\lambda_i X_i) = \lambda_i A_{FX_i} = \lambda_i \lambda_i X_i = \lambda_i^2 X_i$

$$\Rightarrow A_F^2 X_i = \lambda_i^2 X_i$$

By mathematical induction we have, in general we have

$$A_F^m X_i = \lambda_i^m X_i fori = 1, 2, \dots n$$

$\Rightarrow \lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$, where$m$, it is a positive integer, they are fuzzy eigenvalues ofA_F^m

Hence complete the proof

3.2.7. Property – 7

The fuzzy eigenvalues of a fuzzy symmetric matrix are fuzzy numbers and they are real

Proof:

Letλ, it is a fuzzy eigenvalue (it maybe complex) of the fuzzy symmetric matrixA_F

Let the corresponding eigenvector is$X$

LetA_F^T, it is the transpose matrix ofA_F

Since$A_F X = \lambda X$

Pre – multiplying the above equation by $1 \times n$matrix$\overline{X}^T$, where the bar denotes that all elements ofX^T, they are the complex conjugate of thoseX^T, we get

$$\overline{X}^T A_F X = \overline{X}^T \lambda X = \lambda \overline{X}^T X \qquad (12)$$

In (12), taking conjugate complex both sides we get

$$\overline{\left(\overline{X}^T A_F X\right)} = \overline{\left(\lambda \overline{X}^T X\right)} \Rightarrow X^T \overline{A}_F \overline{X} = \overline{\lambda} X^T \overline{X} \qquad (13)$$

SinceA_F, it is a fuzzy matrix that is it has only real elements and therefore$\overline{A}_F = A_F$

$$\Rightarrow X^T A_F \overline{X} = \overline{\lambda} X^T \overline{X} \qquad (14)$$

Again in equation (14) taking transpose both sides we get

$$\left(X^T A_F \overline{X}\right)^T = \left(\overline{\lambda} X^T \overline{X}\right)^T \Rightarrow (X^T)^T A_F^T \overline{X}^T = \overline{\lambda}(X^T)^T \overline{X}^T$$

$\Rightarrow X A_F^T \overline{X}^T = \overline{\lambda} X \overline{X}^T$, since A_F, it is symmetric and then $A_F = A_F^T$ (15)

$\Rightarrow X A_F \overline{X}^T = \overline{\lambda} X \overline{X}^T$

Compare equation (12) and (15) we get

$$\overline{\lambda} X \overline{X}^T = \overline{\lambda} X^T \overline{X}$$

Since $X^T \overline{X}$, it is a 1×1matrix whose elements are only positive value

$\Rightarrow \lambda = \overline{\lambda}$, that is λ , it is real

$\Rightarrow$ The fuzzy eigenvalues of a fuzzy symmetric matrix are fuzzy numbers and they are real

Hence complete the proof

3.2.8. Property – 8

The eigenvectors corresponding to distinct fuzzy eigenvalues of a fuzzy symmetric matrix are orthogonal.

Proof:

Since for any fuzzy symmetric matrix A_F, the eigenvalues are fuzzy and also real numbers (By property – 7). Let X_1, X_2, they are eigenvectors corresponding to two distinct fuzzy eigenvalues λ_1, λ_2. And therefore we have

$A_F X_1 = \lambda_1 X_1$ (16)

$A_F X_2 = \lambda_2 X_2$ (17)

Pre – multiplying (16) by X_2^T, and then we get

$X_2^T A_F X_1 = X_2^T \lambda_1 X_1 = \lambda_1 X_2^T X_1$ (18)

Pre – multiplying (17) by X_1^T, and then we get

$X_1^T A_F X_2 = X_1^T \lambda_2 X_2 = \lambda_2 X_1^T X_2$ (19)

Take transpose both sides of (18) we get

$(X_2^T A_F X_1)^T = (\lambda_1 X_2^T X_1)^T \Rightarrow X_1^T A_F^T X_2 = X_1^T A_F X_2 = \lambda_1 X_1^T X_2$ (Because $A_F^T = A_F$) (20)

But $X_1^T A_F X_2 = \lambda_2 X_1^T X_2$ (21)

$\Rightarrow \lambda_2 X_1^T X_2 = \lambda_1 X_1^T X_2$ By (19) and (21)

$\Rightarrow (\lambda_1 - \lambda_2) X_1^T X_2 = 0$, since $\lambda_1 \neq \lambda_2 \Rightarrow X_1^T X_2 = 0 \Rightarrow X_1, X_2$, they are orthogonal

Hence complete the proof

3.2.9. Property – 9

The similar matrices have same fuzzy eigenvalues

Proof:

LetA_F, B_F, they are two similar fuzzy matrices, and then there exists a non –singular fuzzy matrix$P$, such that

$$B_F = P^{-1}A_F P and \lambda I = P^{-1}\lambda IP$$

$$\Rightarrow B_F - \lambda I = P^{-1}A_F P - \lambda I$$

$$B_F - \lambda I = P^{-1}A_F P - P^{-1}\lambda IP$$

$$\Rightarrow B_F - \lambda I = P^{-1}(A_F - \lambda I)P$$

$$\Rightarrow |B_F - \lambda I| = |P^{-1}||A_F - \lambda I||P|$$

$\Rightarrow |B_F - \lambda I| = |A_F - \lambda I||P^{-1}||P|$(Because determinant product is associative)

$$\Rightarrow |B_F - \lambda I| = |A_F - \lambda I||P^{-1}P| = |A_F - \lambda I||I| = |A_F - \lambda I|$$

$$\Rightarrow |B_F - \lambda I| = |A_F - \lambda I|$$

$\Rightarrow A_F, B_F$, they have the same characteristic polynomials and hence they have same characteristic roots

$\Rightarrow A_F, B_F$, they have the same fuzzy eigenvalues

Hence complete the proof

3.2.10. Property - 10

If a fuzzy symmetric matrix of order 2 has equal fuzzy eigenvalues and then the matrix is a scalar matrix

Proof:

First we use the following rules

Rule – 1:

A Fuzzy symmetric matrix of order$n$, it can always be diagonaized

Rule – 2:

If any diagonal matrix (either fuzzy or classical) with their diagonal elements is equal then the diagonal matrix is a scalar matrix

Given that a fuzzy symmetric matrix of order 2 has equal fuzzy eigenvalues

By rule – 1, we get

For any fuzzy symmetric matrixA_F, it can always be diagonalized

Let$\lambda_1 and \lambda_2$, they are their fuzzy eigenvalues and then we get the diagonalized matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = A_F$$

Given that$\lambda_1 = \lambda_2 \Longrightarrow \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} = A_F$, by rule – 2 we get

A_F, it is a scalar matrix

Hence complete the proof

3.2.11. Property – 11

The eigenvector$X$, of fuzzy matrixA_F, it is not unique

Proof:

Letλ, it is the fuzzy eigenvalue ofA_F, and then the corresponding eigenvector$X$ such that

$A_F X = \lambda X$ (22)

Multiplying both sides of (22) by a non –zero scalar$k$, we get

$k(A_F X) = k(\lambda X) \Longrightarrow A_F(kX) = \lambda(kX)$(Because$k$, it is a non – singular scalar)

$$\Longrightarrow A_F = \lambda$$

That is eigenvector is determined by a multiplicative scalar

$\Longrightarrow$ The eigenvector$X$, of fuzzy matrixA_F, it is not unique

Hence complete the proof

3.2.12. Property – 12

If$A_F and B_F$, they are$n \times n$ fuzzy matrices andB_F, it is non singular fuzzy matrix and then$A_F and B_F^{-1} A_F B_F$, they have the same eigenvalues

Proof:

Since the characteristic polynomial of$B_F^{-1} A_F B_F$, it is

$$|B_F^{-1} A_F B_F - \lambda I| = |B_F^{-1} A_F B_F - \lambda B_F^{-1} B_F| = |B_F^{-1} A_F B_F - B_F^{-1} \lambda B_F|$$

(Sinceλ, it is a constant)

$$\Longrightarrow |B_F^{-1} A_F B_F - \lambda I| = |B_F^{-1}(A_F - \lambda) B_F| = |B_F^{-1}(A_F - \lambda I) B_F|$$

$$\Longrightarrow |B_F^{-1} A_F B_F - \lambda I| = |B_F^{-1}||A_F - \lambda I||B_F|$$

$\Longrightarrow |B_F^{-1} A_F B_F - \lambda I| = |B_F^{-1}||B_F||A_F - \lambda I|$ (Since determinant value is constant)

$\Longrightarrow |B_F^{-1} A_F B_F - \lambda I| = |B_F^{-1} B_F||A_F - \lambda I| = |I||A_F - \lambda I| = |A_F - \lambda I|$ (Since$|I| = 1$)

$\Longrightarrow$Characteristic polynomial of$B_F^{-1} A_F B_F$ =Characteristic polynomial ofA_F

$\Longrightarrow A_F and B_F^{-1} A_F B_F$, they have the same eigenvalues

Hence completes the proof.

4. CONCLUSION

In this article we proved some properties of eigenvalues and eigenvectors of classical matrices, applicable in fuzzy matrices. Since fuzzy matrix topic is vast area, and also we can try to apply the properties of eigenvalues and eigenvectors of fuzzy matrix in Heat transfer equations, Optimal Control theory, Sensitive analysis, Quantum and Dynamic Mechanics, etc., If any one interest to do this work continue, they may try to prove eigenvalues and eigenvectors of fuzzy matrix concept in Orthogonality, spectral theory, etc., with some relevant definition and basic concepts of classical matrices.

REFERENCES

1. Bellman, R. E. Zadeh, L. A. Decision making in a fuzzy environment management science, 17., pp: 141 – 164, 1965
2. Buslowwick, M. Inversion of characteristic matrix of the time delay systems of neural type, Foundation of Control Engineering Vol. 9. No. 4, pp: 195 – 201, 1984
3. Buslowwick, M., Kaczorek, T. Reachability and minimum energy control of positive linear discrete time systems with delay. Proc. 12th Mediterranean Conf. on Control and Automation, Kasadai – Izmir, Turkey, 2004
4. Chang F. R., Chan C. N., The generalized Cayley – Hamilton System ad Control Let. 18, pp: 179 – 182, 1992.
5. Das, S. Chakaraverty, Fuzzy system of linear equations applied in circuit analysis. International journal of computer and electrical engineering Vol. 1, No. 5, pp: 537 – 538, 2009
6. Dehaghan, M., Hashemi, Ghatee, M. Computational methods for solving fully fuzzy linear systems, Applied mathematics and computation 179,pp: 328 – 343, 2006. http://dx.doi.org/10.1016/j.amc.2006.05.043
7. Kazorek, T. Vectors and matrices in Automation and Electrotechnics, WNT Warszawa (in Polish) 1988.
8. Kazorek, T. Extension of the Cayley – Hamilton Theorem for 2D continuous – discrete linear Systems, Appl. Math. And Computer Science, Vol. 4. No.4. Pp: 507 – 515

9. Kazorek, T. Extension of the Cayley – Hamilton Theorem for singular 2D linear systems with non – square matrices. Bull. Pol. Acad. Technology Science Vol. 43, No. pp: 39 – 48, 1995

9b. Kazorek, T. An extension of the Cayley – Hamilton Theorem for a standard pair of block matrices, Appl. Math, And Computer Science Vol. 8 No. 3, pp: 511 – 516, 1998

10. Lotfi Zadeh, A. Fuzzy sets, information and control, 8: pp: 338 – 353, 1965.

11. Lewis, F. L., Further remarks on the Cayley – Hamilton Theorem and Fadeev's method for the matrix pencil, IEEE Trans. Automat. Control, 31, pp: 869 – 870, 1986.

12. Michael Hans, "Applied fuzzy Arithmetic" Part – I, pp: 41 – 78, 1005

13. Zimmermann, H.J. Fuzzy set theory and its applications, Second Edition, 1996.