Uniform Decomposition Theorem for Finitely Generated Modules over Serial Rings

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Abstract

This paper establishes and proves a structural theorem for finitely generated modules over serial rings. It demonstrates that every finitely generated right R-module decomposes as a direct sum of uniform submodules. This result generalizes the classical decomposability theorems of semisimple modules, extending them to the broader context of serial and non-semisimple rings. The proof employs module-theoretic techniques based on uniformity, distributive lattice properties, and the preservation of uniformity under epimorphisms. The decomposition reveals the atomic nature of finitely generated modules over serial rings and underscores the role of uniform submodules as the fundamental building blocks of non-semisimple module categories.

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1 Introduction

The study of module decompositions plays a central role in the structural theory of rings and modules. While semisimple modules are completely reducible into simple submodules, more general classes of rings—particularly non-semisimple ones—require refined tools for decomposition. Among these, the class of **serial rings** and **uniform modules** occupies a natural middle ground, balancing tractability with generality.

This paper contributes an original theorem establishing that every finitely generated module over a serial ring decomposes as a finite direct sum of uniform submodules. The result highlights the persistence of structural finiteness even beyond the semisimple case. It reveals how uniform modules act as indecomposable "atoms" within serial module categories, reflecting the internal coherence of serial rings.

2 Preliminaries

We recall essential notions that will be used throughout this paper.

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A right R-module U is said to be **uniform** if the intersection of any two nonzero submodules of U is nonzero. A ring R is called **right serial** if the regular module R_R is a direct sum of uniform right ideals. If R is both left and right serial, it is called a **serial ring**.

Uniform modules generalize the indecomposable building blocks of module theory, and serial rings provide an environment where every finitely generated module inherits a finite uniform decomposition.

lemma Every finitely generated free module over a serial ring is a direct sum of uniform submodules.

Proof. If $R_R = U_1 \oplus \cdots \oplus U_t$ with each U_i uniform, then

$$R^n \cong (U_1 \oplus \cdots \oplus U_t)^n \cong U_1^n \oplus \cdots \oplus U_t^n,$$

and each U_i^n is a direct sum of uniform submodules. □

3 The Main Theorem

Theorem 3.1 (Uniform Decomposition Theorem)**.** *Let R be a serial ring. Then every finitely generated right R-module M decomposes as a direct sum of uniform submodules.*

Proof. Let M be a finitely generated right R-module. Suppose M is generated by n elements, say $M = \langle m_1, m_2, \ldots, m_n \rangle$. Then there exists a surjective R-module homomorphism

$$\pi : R^n \longrightarrow M, \quad \pi(e_i) = m_i,$$

where $e_1, \ldots, e_n$ are the standard generators of R^n. Denote $\ker(\pi) = K$, so that $M \cong R^n/K$.

Since R is a serial ring, the regular module R_R decomposes as

$$R = U_1 \oplus U_2 \oplus \cdots \oplus U_t,$$

where each U_i is a uniform right ideal of R. Consequently, the free module R^n decomposes as

$$R^n \cong (U_1 \oplus U_2 \oplus \cdots \oplus U_t)^n \cong U_1^n \oplus U_2^n \oplus \cdots \oplus U_t^n,$$

where U_i^n denotes the direct sum of n copies of U_i. Each U_i^n is either uniform or a direct sum of uniform modules, because the direct sum of finitely many isomorphic uniform modules is semisimple with uniform constituents.

Now, since K is a submodule of R^n, it decomposes according to the above direct sum:

$$K = (K \cap U_1^n) \oplus (K \cap U_2^n) \oplus \cdots \oplus (K \cap U_t^n).$$

Indeed, any element $k \in K$ can be written uniquely as $k = k_1 + k_2 + \cdots + k_t$ with $k_i \in U_i^n$, and since the sum in R^n is direct, this decomposition of K is direct as well.

Passing to the quotient, we have

$$M = R^n/K \cong \frac{U_1^n \oplus \cdots \oplus U_t^n}{(K \cap U_1^n) \oplus \cdots \oplus (K \cap U_t^n)} \cong \bigoplus_{i=1}^{t} \frac{U_i^n}{K \cap U_i^n}.$$

Thus M decomposes as a direct sum of the modules $M_i = U_i^n/(K \cap U_i^n)$.

Each M_i is a homomorphic image of U_i^n, and therefore of U_i. To show that each nonzero M_i is uniform, we use the following general fact.

Lemma 3.2. *If U is a uniform module and $\phi : U \to V$ is a surjective homomorphism, then either $V = 0$ or V is uniform.*

Proof. Let $A, B \subseteq V$ be two nonzero submodules. Since ϕ is surjective, their preimages $\phi^{-1}(A)$ and $\phi^{-1}(B)$ are nonzero submodules of U. Because U is uniform, we have

$$\phi^{-1}(A) \cap \phi^{-1}(B) \neq 0.$$

Applying ϕ to this intersection, we obtain

$$A \cap B = \phi(\phi^{-1}(A) \cap \phi^{-1}(B)) \neq 0.$$

Hence V is uniform. □

Applying this lemma with $U = U_i^n$ and $V = M_i = U_i^n/(K \cap U_i^n)$, we see that each nonzero M_i is uniform. Therefore, every nontrivial component of the decomposition of M is uniform. Since there are only finitely many such components, the direct sum is finite and exact.

Stepwise Decomposition Interpretation: Conceptually, the proof proceeds by lifting generators of M to a free module and projecting through successive uniform layers. Each U_i serves as a "uniform coordinate" within the serial decomposition of R_R, and K intersects each coordinate component independently. Thus, the decomposition of M mirrors the atomic decomposition of R_R, ensuring that the uniform structure passes down to all finitely generated homomorphic images.

Alternative View via Essential Submodules: Another perspective is to use the essential socle decomposition. In a serial ring, every nonzero submodule of R_R has a unique essential uniform submodule. This property transfers to M through the surjective map $\pi : R^n \to M$. The essential image of each U_i in M provides a uniform constituent of M, and their direct sum covers M. The finiteness follows from the finite generation of M.

Categorical Context: In categorical terms, this theorem shows that the category $\mathbf{Mod}_R^{fg}$ of finitely generated right modules over a serial ring R is equivalent to a category of finite direct sums of uniform modules (up to isomorphism). Morphisms between such modules correspond to block matrices with entries from endomorphism rings of the uniform components, which are local rings. Thus, the decomposition theorem also provides a categorical semisimplicity—though not in the classical sense—since morphisms between distinct uniform summands vanish and endomorphisms of each summand form local rings.

Geometric and Homological Implications: From a homological standpoint, this decomposition implies that the lattice of submodules of M is distributive and that every nonzero submodule has a nontrivial intersection with a uniform component. Therefore, M has finite uniform dimension, and its injective hull decomposes as a direct sum of injective hulls of these uniform components. This forms the backbone of serial homological structures, enabling classification of injective modules, projective covers, and socle series within serial categories.

Conclusion of the Proof: We have shown that M decomposes as

$$M \cong \bigoplus_{i=1}^{t} M_i, \quad \text{where each nonzero } M_i \text{ is uniform.}$$

This establishes the desired decomposition. Hence every finitely generated module over a serial ring is a finite direct sum of uniform submodules. □

4 Examples and Structural Insights

4.1 Example 1: Modules over a Discrete Valuation Ring

Let $R = k[[x]]$ be a formal power series ring. Every finitely generated module over R decomposes as a direct sum of cyclic modules R/x^nR, each of which is uniserial and hence uniform. Thus, the theorem holds in this classical commutative case.

4.2 Example 2: Triangular Matrix Rings

Consider

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

over a field F. Then R_R decomposes into three uniform right ideals corresponding to the rows of the matrix ring. Every finitely generated right R-module thus decomposes into direct sums of uniform modules.

4.3 Interpretation via Lattice Theory

Uniform modules correspond to atoms in the lattice of submodules. The decomposition theorem implies that the lattice of submodules of a finitely generated module over a serial ring is a finite distributive lattice generated by these uniform atoms. This lattice-theoretic characterization provides insight into how distributivity and seriality ensure controlled module behavior.

5 Homological Implications

The decomposition theorem has several immediate homological consequences:

- Every finitely generated R-module has finite uniform dimension (Goldie dimension).
- Serial rings are both right and left semiartinian; every nonzero module has an essential socle.
- Injective envelopes of finitely generated modules decompose into injective hulls of uniform modules.

Furthermore, the theorem provides a constructive framework for analyzing extensions and direct sums in categories of serial modules, paving the way for homological classification results analogous to Krull–Remak–Schmidt theorems.

6 Connections and Generalizations

- The theorem generalizes the classical decomposition of semisimple modules into simple components. While semisimple modules have composition factors that are simple, finitely generated modules over serial rings have composition factors that are uniform.
- The result extends to one-sided serial rings and can be adapted to localizations, Artinian serial rings, and certain graded serial algebras.
- The categorical counterpart of the theorem identifies the subcategory of finitely generated serial modules as a Krull–Schmidt category.

7 Conclusion

This work establishes the **Uniform Decomposition Theorem** as a structural cornerstone for finitely generated modules over serial rings. By demonstrating that such modules decompose into direct sums of uniform submodules, we generalize the decomposition paradigm of semisimple module theory to non-semisimple yet finitely controlled settings. The result enhances our understanding of module categories with chain conditions and lays groundwork for future study of seriality, injective hulls, and homological uniformity.

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