COUPLED FIXED POINT THEOREM IN COMPLETE PARTIALLY ORDERED METRIC SPACE

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ABSTRACT

In this paper, we have proved a coupled fixed point theorem in partially ordered metric spaces by employing a contractive condition. Our results generalized and extent the work of Bhaskar and Lakshikhantham [1].

Keywords: Coupled fixed point theorems, partially ordered metric space.

200 AMS Subject Classification: 47H10, 54H25, 47H09.

§1. INTRODUCTION

The notion of coupled fixed pint was introduced by Chang and Ma [2]. Since the, the concept has been of interest to many researchers in metrical fixed pointtheory. Bhaskar and Lakshimikantham [1] established coupled fixed point theorems in a metric space endowed with partial order by employing contractive condition. Harjani et al. [4] established some fixed point in partially ordered metric space by using contractive condition of rational type. Motivated by works of Bhaskar [1] and Harjani [4], in the present paper we have proved a coupled fixed point theorem in partially ordered metric space by employing some notion of Bhaskar [1] and Harjani [4], in the present paper, we have proved a coupled fixed point theorem in partially ordered metric space by employing some notions of Bhaskar and Lakshmikantham [1] as well as a rational type contrction. The result of our theorem generalised and extended the work of Bhaskar and Laxminathan [1].

Let (X,d) be a metric space. An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping $f : X \times X \to X$ if

$$f(x,y) = x \text{ and } f(y,x) = y$$

Suppose $(X,\preceq)$ be a partially ordered set and $f : X \times X \to X$ we say that f has the mixed monotone property if $f(x,y)$ is monotone nondecreasing in x and monotone nonincreasing in y, that is $\forall x,y \in X$,

$$\forall x_1,x_2 \in X,\ x_1 \preceq x_2 \Rightarrow f(x_1,y) \preceq f(x_2,y)$$

and

$$\forall y_1,y_2 \in X,\ y_1 \preceq y_2 \Rightarrow f(x,y_1) \succeq f(x,y_2)$$

Definition 1. Let $(X,\leq)$ be a partially ordered set and $F: X \times X \to X$. The mapping F is said to have the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone non increasing in its second argument, that is for all $x_1,x_2 \in X$, $x_1 \leq x_2$ implies

$F(x_1,y) \leq F(x_2,y)$ for any $y \in X$,

and for all $y_1,y_2 \in X$, $y_1 \leq y_2$ implies

$F(x_1,y) \geq F(x_2,y)$ for any $x \in X$,

Theorem 1. Let $(X,\preceq,d)$ be a partially ordered complete metric space. suppose $f: X \times X \to X$ be a continuous mapping which has the mixed monotone property such that

$$d(f(x,y),f(u,v) \preceq a\{d(x,u)+d(u,f(x,y))\}$$

$$+bd(x,f(x,y)+cd(u,f(u,v)) \quad (3.3)$$

for every two pairs of points $(x,y),(u,v) \in X \times X$ such that $x \neq u$ and

$\frac{c}{1-(a+b)} \in (0,1)$.

Proof. Choose an arbitrary pair $(x_0,y_0) \in X \times X$ and set $x_1 = f(x_0,y_0), y_1 = f(y_0,x_0)$ and we can choose $x_2,y_2 \in X$ such that $x_2 = f(x_1,y_1)$ and $y_2 = f(y_1,x_1)$ therefore

$$f^2(x_0,y_0) = f(f(x_0,y_0),f(y_0,x_0))$$

$$= f(x_1,y_1) = x_2$$

and $$f^2(y_0,x_0) = f(f(y_0,x_0), f(x_0,y_0))$$
$$= f(y_1,x_1) = y_2$$

Due to mixed monotone property of f we obtain,

$$x_2 = f^2(x_0,y_0) = f(x_1,y_1) \geq f(x_0,y_0) = x_1$$

$$y_2 = f^2(y_0,x_0) = f(y_1,x_1) \leq f(y_0,x_0) = y_1$$

In general, we have that for $n \in N$,

$$x_{n+1} = f^{n+1}(x_0,y_0) = f(f^n(x_0,y_0), f^n(y_0,x_0))$$

$$y_{n+1} = f^{n+1}(x_0,y_0) = f(f^n(y_0,x_0), f^n(x_0,y_0))$$

$$d(x_{n+1},x_n) = d(f(x_n,y_n), f(x_{n-1},y_{n-1}))$$

$$\leq a\{d(x_n,x_{n-1}) + d(x_{n-1}, f(x_n,y_n))\}$$

$$+bd(x_n, f(x_n,y_n)) + cd(x_{n-1}, f(x_{n-1},y_{n-1}))$$

$$\leq a\{d(x_n,x_{n-1}) + d(x_{n-1},x_{n+1})\} + bd(x_n,x_{n+1}) + cd(x_{n-1},x_n)$$

$$\leq (a+b)d(x_n,x_{n+1}) + cd(x_{n-1},x_n)$$

$$d(x_{n+1},x_n) \leq \frac{c}{1-(a+b)} d(x_{n-1},x_n)$$

Similarly we have

$$d(y_{n+1},y_n) = d(f(y_n,x_n), f(y_{n-1},x_{n-1}))$$

$$d(x_n,x_{n+1}) + d(y_n,y_{n+1}) \leq \frac{c}{1-(a+b)} \left[d(x_n,x_{n-1}) + d(y_n,y_{n-1})\right]$$

Let $d_n = d(x_n,x_{n+1}) + d(y_n,y_{n+1})$ $\lambda = \dfrac{c}{1-(a+b)}$.

Then we have,

$$d_n \leq \lambda d_{n-1} \leq \lambda^2 d_{n-2} \leq \cdots \leq \lambda^n d_0$$

If $d_0 = 0$, then (x_0,y_0) is a coupled fixed point of f.

Suppose that $d_0 > 0$, Then for each $r \in N$. We obtain by triangle inequality

$$d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \leq [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$$

$$+ \cdots d(x_{n+r-1}, x_{n+r})]$$

$$\leq d_n + d_{n+1} + \cdots d_{n+r-1}$$

$$\leq \frac{\lambda^n (1-\lambda^r) d_0}{1-\lambda} \to 0, \qquad n \to \infty$$

therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,d). Since (X,d) is a complete metric space, there exists $x^*, y^* \in X$ such that

$$\lim_{n \to \infty} x_n = x^*$$

and $\lim_{n \to \infty} y_n = y^*$ we now show that (x^*, y^*) is a coupled fixed point of f

Let $\varepsilon > 0$ continuity of f at (x^*, y^*) implies that for a given $\frac{\varepsilon}{2} > 0$, there exist a $\delta > 0$ such that $d(x^*, u), d(y^*, v) < \delta$ implies

$$d(f(x^*, y^*), f(u, v) < \frac{\varepsilon}{2}$$

$\{x_n\} \to x$ and $\{y_n\} \to y$ for $k = \min\left(\frac{\varepsilon}{2}, \frac{\delta}{2}\right) > 0$,

there exist n_0, m_0, such that for $n \geq n_0$, $m \geq m_0$ we have $d(x_n, x^*) \leq k$ and $d(x_m, x^*) \leq k$ therefore for $n \in N$, $n \geq \max\{n_0, m_0\}$

$$d(f(x^*, y^*), x^*) \leq d(f(x^* y^*), x_{n+1}) + d(x_{n+1}, x^*)$$

$$= d(f(x^*, y^*), f(x_n, y_n)) + d(x_{n+1}, x^*) \leq \frac{\varepsilon}{2} + k \leq \varepsilon$$

from which it follows that

$$f(x^*, y^*) = x^*.$$

In a similar manner we can show that $f(y^*, x^*) = y^*$.

Hence (x^*, y^*) is a coupled fixed point of f.

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