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SOLVING EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR CLASS OF LINEAR ELLIPTIC EQUATIONS OF DIRICHLET PROBLEM IN DIVERGENCE FORM

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Abstract: *In this article we are concerned with Existence and Uniqueness of Solutions for a Class of Linear Elliptic Equations of Dirichlet Problem in Divergence Form. We obtain Weak Solutions from classical solution for equations by relaxing the Conditions on the solution and on the data f of the given problem.*

$L(x,D)u = \sum_{0\leq|\sigma|,|\gamma|\leq k}(-1)^{|\sigma|} D^{|\sigma|}\left(a_{\sigma\gamma}(x)D^{\gamma}u\right) = f \quad in \;\Omega$ *, and*

$D^{\alpha}u = 0 \quad for\; |\alpha| \leq k-1 \quad on\; \partial\Omega$

Key Words: *Sobolev spaces, Elliptical point or Domain, Garding's Inequality, Class Linear Functions, Support*

INTRODUCTION

In this paper, we are concerned with the following elliptic problem:

$$L(x,D)u = \sum_{0\leq|\sigma|,|\gamma|\leq k}(-1)^{|\sigma|} D^{|\sigma|}\left(a_{\sigma\gamma}(x)D^{\gamma}u\right) = f \quad in \;\Omega \longrightarrow (1)$$

And $D^{\alpha}u = 0 \quad for\; |\alpha| \leq k-1 \quad on\; \partial\Omega \longrightarrow$ (2)

Basic Concepts and Some Definitions:

Definition for Soboleve space: [1], [12]

The Soboleve space $W^{k,p}(\Omega)$ consists of those functions $f \; in \; L^p(\Omega)$ such that all the distributional derivatives of f of order at least k and they are also in $L^p(\Omega)$, or

$W^{k,p}(\Omega) = \{f \in L^p(\Omega) | D^{\alpha}f \in L^p(\Omega) \;\; for\; all\; |\alpha| \leq k\}$

Definition for Partial Differential Equation:

A partial deferential equation is an equation involving an unknown function of two or more variables and certain of its partial derivatives and an expression of the form is

$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), x) = 0 \;\; \forall \;\; x \in \;\Omega$

It is called a k^{th} order partial differential equation where

$F: \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$, it is given and

$u: \Omega \longrightarrow \mathbb{R}$, it is the unknown function. The Soboleve spaces are useful tools to solve partial differential equation.

Remark:

There is an important notation "Well – posed problems" in Partial Differential Equation Theory (PDET). Roughly a given problem of PDE is well posed if

a. The problem has in fact a solution
b. The exists solution is unique
c. Exists solution depends continuously on the data given in the problem.

The condition (c) of particular importance in its application of problem arising from Differential Geometry and Physics:

Basic Theorems and Definitions:

Definition for Class Functions:

Let $p \geq 1$ it is a real number, Ω it is an open subset of $\mathbb{R}^n$ $for\ n \geq 1$, and then

$L^p(\Omega) = \{$Class of functions$v: \Omega \longrightarrow \mathbb{R}$ $such\ that\ |v(x)| \leq C\ almost\ everywhere\ x \in \Omega\}$

Equipped with the norms

$$\|v\|_{p,\Omega} = \left(\int_{\Omega} |v(x)|^p\, dx\right)^{\frac{1}{p}} \longrightarrow \quad (1.1)$$

And $\|v\|_{\infty,\Omega} = \inf\{C, such\ that\ |v(x)| \leq C\ almost\ everywhere\ x \in \Omega\} \longrightarrow (1.2)$

Moreover for any $1 \leq p < \infty$ the dual of $L^p(\Omega)$(The set of all bounded linear functional) can be defined with $L^q(\Omega)$ $where\ q$, it is the conjugate number of p such that $\frac{1}{p} + \frac{1}{q} = 1$

Definitions:

1. The **class of local function** is that, the set of functions v defined on Ω such that for any $\overline{\Omega'} \subset \Omega$ bounded such that one has $v \in L^q(\Omega')$included and denoted by $L^p_{loc}(\Omega')$
2. The **support** of a function f, it is defined as $supp\ f$ =The closure of the set $\{x \in \Omega / fx \neq 0$and

   $supp\ f = \overline{\{x \in \Omega | f(x) \neq 0\}}$
3. The **space of functions** infinitely differentiable in with compact support in Ω and denoted by $D(\Omega)$

Definition for Weak or Distributional:

Let α, it is a multi index and supposes that $u, v \in L^1_{loc}(\Omega)$ and $\int_{\Omega} u(x)\, D^{\alpha}_{\omega}\eta(x)dx = (-1)^{|\alpha|} \int_{\Omega} v(x)\, \eta(x)\, dx\ \ \forall\, \eta \in C_0^{\infty}(\Omega)$

Then v, it is called the weak or distribution partial derivative of $u \ in \ \Omega$, and it is denoted by $D_\omega^\alpha u$

Notes:

a. If $u(x)$ sufficiently smooth to have continuous derivative of $D_\omega^\alpha u$, by integrating use integrating by parts $\int_\Omega^\cdot u(x)\, D_\omega^\alpha \eta(x) dx = (-1)^{|\alpha|} \int_\Omega^\cdot v(x)\, \eta(x)\, dx$, and hence the classical derivative ∂^α, it is also the weak derivative and also $D_\omega^\alpha u$ it may exist in the weak sense without existing in the classical sense.

b. In the soboleve space $W^{k,p}(\Omega)$, let $u \in L^p(\Omega)\ and\ u \in W^{k,p}(\Omega)$, if for any multi index α such that $|\alpha| \leq k, D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$

c. If $p = 1$, the notation $W^{k,p}(\Omega)$, becomes $W^{k,1}(\Omega)$, and if $p = 2$, the notation $H^k(\Omega)$ it is used as $H^k(\Omega) = W^{k,2}(\Omega)$ $\longrightarrow$ (1.3)

d. And also the letter H certainly stands for H^k, it is a Hilbert Space

e. We also certainly define $W_{loc}^{k,p}(\Omega)$ as in the definition of $L_{loc}^p(\Omega)$ that is we define the norm of $W^{k,p}(\Omega),\ as\ 1 \leq p < \infty$

f. $\|u\|_{k,p}^p = \|u\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \int_\Omega^\cdot |D_\alpha u|^p\, dx = \sum_{|\alpha| \leq k} \|D_\alpha u\|_p^p$ $\longrightarrow$ (1.4)

g. And for $p = 2$, we define an inner product by

$\langle u, v \rangle_k = \sum_{|\alpha| \leq k} \int_\Omega^\cdot D_\alpha \overline{u(x)}\, v(x)\, dx$ $\longrightarrow$ (1.5)

Examples:

1. For $1 \leq p < \infty$, we have

$$\|u\|_{1,p} = \left\{\int_\Omega^\cdot |u|^p + \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx\right\}^{\frac{1}{p}}$$

$$\|u\|_{2,p} = \left\{\int_\Omega^\cdot |u|^p + \sum_{i=1}^m \left|\frac{\partial u}{\partial x_i}\right|^p dx + \sum_{i,j=1}^m \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^p dx\right\}^{\frac{1}{p}}$$, and so on

$$W_0^p = \{u \in W^{k,p} | u_k \longrightarrow u \in W^{k,p}(\Omega), for\ some\ sequence\ (u_k) \in C_0^\infty(\Omega)\}$$

2. Let Ω it is an open subset of $\mathbb{R}^n$, and denote by $H^1(\Omega)$, it is the subset of $L^2(\Omega)$, and it is defined by $H^1(\Omega) = \{v \in L^2(\Omega) | \partial_{x_i} v \in L^2(\Omega), for\ all\ i = 1, \dots, n\}$, where $\partial_{x_i} v$, it is the derivative of distribution sense.

3. In what follows we will be in deed of functions vanishing on the boundary $\partial\Omega\ of\ \Omega$. However for a class of functions in $L^2(\Omega)$, the meaning of its value on $\partial\Omega$, it is not clear. So we will overcome this problem by introducing

$$H_0^1(\Omega) = \{The\ closure\ of D(\Omega) \in H^1(\Omega)\} = \left\{\overline{D(\Omega)} \in H^1(\Omega)\right\}$$

4. The closure being understood for the norm (1.4) and also $H_0^1(\Omega)$, it will play the role of the function $H^1(\Omega)$, and it vanish on $\partial\Omega$
5. The dual space of $H_0^k(\Omega)$, and it is denoted by $H^{-k}(\Omega)$, i.e.,

$$H^{-k}(\Omega) = \left(H_0^k(\Omega)\right)^* \longrightarrow \quad (1.6)$$

Theorem – 1:

If $f \in L_{loc}^1(\Omega)$ such that $D^\alpha f$ exists, for every $\alpha, |\alpha| \le k$ and if $g \in C^k(\Omega)$, and then fg admits all weak derivatives up to order k and also

$$D^\alpha(fg) = \sum_{|\alpha|\le k} \binom{\alpha}{\beta} D^\beta g D^{\alpha-\beta} f \longrightarrow \quad (1.7)$$

Where $\alpha > \beta$ and $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$

Theorem – 2 – (Holder's Inequality) [9], [13], [11]

If f and g, they are measurable functions defined on Ω and p and q they are conjugate and then

$$\int_\Omega |fg|\, d\mu = \|fg\|_1 \le \|f\|_p \|g\|_p \longrightarrow \quad (1.8)$$

Definition for $\tilde{L}_\omega^2(\Omega)$:

Let Ω it is an open subset of $\mathbb{R}^n$, and let $\omega: \Omega \longrightarrow \mathbb{R}^+$, it is a continuous function and then define

$$\tilde{L}_\omega^2(\Omega) = \left\{u \in C(\overline{\Omega}) \middle| \int_\Omega \omega(x)|u(x)|^2 dx < \infty\right\} \longrightarrow \quad (1.9)$$

And also the inner product is defined by $\langle u, v\rangle = \int_\Omega \omega(x)\overline{|u(x)|} v(x) dx$

The space $L_\omega^2(\Omega)$, it is the completion of $\tilde{L}_\omega^2(\Omega)$

Theorem – 3:

The Fourier transform F it is a homeomorphism from $H^k(\mathbb{R}^n)$ onto the weighted space $L_\omega^2(\mathbb{R}^n)$ where $\omega(\varepsilon) = 1 + |\varepsilon|^{2k}$ and denotes L_k^2, it is weighted L^2 space implies that $D(\mathbb{R}^n)$, it is the dense in $H^k(\mathbb{R}^n)$

Definition for Embedded:

Let Y, Z, they are Banach spaces, and say Y, it is continuously embedded in Z and writes [3] $Y \hookrightarrow Z, and\ if\ Y \subset Z$, and then there is a constant C such that

$$\|u\|_Z \le C\|u\|_Y, \forall\ u \in Y \longrightarrow \quad (1.10)$$

Definition for Compactly Embedded:

Let X, Y, they are Banach spaces such that X, it is continuously embedded in Y and say that X, it is compactly embedded in Y, and denoted by $X \hookrightarrow^C Y$, and also if the unit ball in X, it is pre compact in Y, or, equivalently, every bounded sequence in X, it has a sub sequence that convergence in Y

Lemma – 1 – (Ehrling's Lemma): [8]

Let X, Y, Z, they are Banach spaces. Assumes that X, it is continuously embedded in Y *and* Y, it is continuously embedded in Z, i.e. $X \hookrightarrow^C Y \hookrightarrow Z$, and then, for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that $\|x\|_Y \leq \varepsilon\|x\|_X + \|x\|_Z, \forall\ x \in X$

Remark:

Assume that it is bounded and $H^k(\Omega) \hookrightarrow^C H^{k-1}(\Omega)$ then the following norms on H^k, they are equivalent

$$\|u\|_{k,2}^2 = \sum_{|\alpha|\leq k}\|D^\alpha u\|_2^2 \longrightarrow \quad (1.11)$$

$$\|u\|_{12,k,*}^2 = \sum_{|\alpha|=k}\|D^\alpha u\|_2^2 + \|u\|_1^2 \longrightarrow \quad (1.12)$$

And also by the Poincare's Inequality we can easily verify that in the space H_0^k, we simply leave the norm $\|u\|_1^2$, and also Ω, it is need not be bounded, it suffices that it to be bounded in one direction

Theorem – 4 – Poincare's Inequality [2], [3], [4], [8], [11]

Let Ω it is contained in the strip $|x_1| \leq d < \infty$. Then there is a constant C, depending only on k *and* d, such that $\|u\|_{k,2}^2 \leq C\sum_{|\alpha|=k}\|D^\alpha u\|_2^2,\ u \in H_0^k \longrightarrow$ (1.13)

Remark:

For $p \in [1, \infty)$, the above result holds. In other words satisfies the hypothesis of theorem – 4 and then there is a constant C, depending only on $k\ \ d$, *and* p, such that

$$\|u\|_{k,2}^2 \leq C\sum_{|\alpha|=k}\|D^\alpha u\|_2^2,\ u \in W_0^{k,p} \longrightarrow \quad (1.14)$$

Notations Used in this Article:

Let Ω, it is an open subset of $\mathbb{R}^n$, and $u: \Omega \longrightarrow \mathbb{R}$ *and* Ω, it is a function and Ω it is assumed to be connected bounded if not specified otherwise: Without specification, we always assume that u, it has enough derivatives and we use the following notation for derivatives:

$Du = (D_1u, D_2u, \dots, D_nu);\ D_iu = \frac{\partial u}{\partial x_i}; D_{ij}\,u = \frac{\partial^2 u}{\partial x_i \partial x_j}$, and also

$\Delta u = \sum D_{ii}u = 0$, and it is called **The General Laplace Equation**.

We also use multi index notations for further research in this article as follows:

a. A vector of the form $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *where* $\alpha_i \in N_0^n$, it is called the multi index of the order $|\alpha| = \sum_{i=1}^n \alpha_i$

b. For a given a multi index α, define $D^\alpha u(x) = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha 1} \dots \partial x_n^{\alpha n}}$

c. For any $m \in N$, denoted $D^m u(x) = \{D^\alpha u(x) | |\alpha| = m\}$, the set of all partial derivatives of order m, and also regard $D^m u(x)$ as a point in $\mathbb{R}^{n^m}$ (with any specified order) and

$| D^m u(x)|^2 = \sum_{|\alpha|=m} | D^\alpha u(x)|^2$

When $m = 1$ *then* $Du = (D_1 u, D_2 u, \dots, D_n u)$, it is the **Gradient operator** that is

$|\nabla u| = \left(\sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} |D_j u|^2\right)^{\frac{1}{2}}$

d. When $m = 2$ *then* $D^2 u = D_{ij} u$ *as* $n \times n$ matrix and it is called **Hessian** of u

Linear Elliptic Equation: [8]

The symbol of the expression $L(x, D)$, it is given by $L(x, i\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)(i\xi)^\alpha$, and also the principal part of the symbol is $L^p(x, i\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha$

Example:

Consider the second order PDE in two space dimensions

$Lu = a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$

The principal part of the symbol of L, it is $L^p(x, y, i\xi, i\eta) = -a(x,y)\xi^2 - b(x,y)\xi\eta - c(x,y)\eta^2$

This can be represented in matrix form as $L^p = [\xi \quad \eta] \begin{bmatrix} -a(x,y) & -\frac{1}{2}b(x,y) \\ -\frac{1}{2}b(x,y) & -c(x,y) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$

Definition for Elliptic at a Point:

A differential operator of order m, it is elliptic at x_0, if and only if

$L^p(x_0, \xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha \neq 0, \forall\, \xi \in \mathbb{R}^n/\{0\}$

Lemma – 2:

If a linear partial differential operator L of order m, it is elliptic at $x_0 \in \mathbb{R}^n, for\ n > 1$ then m, it is an even integer $(m = 2k)$ *and* $\xi \longrightarrow L^p(x_0, \xi)$, it is continuous and takes on the value 0, only at $\xi \neq 0$

Proof:

By the definition assume that, $\xi \longrightarrow L^p(x_0, \xi)$ it is continuous and takes on the value 0, only at $\xi = 0$

Suppose $L^p(x_0, \xi) < 0$ *and* $L^p(x_0, \xi) > 0$, and then connect ξ_1 *and* ξ_2, using a path not going through 0, as noted: $L^p(x_0, \xi)$ it must vary continuously along the path, taking on the value 0, this is contradiction to our assumption

It now follows that, for any $\xi \in \mathbb{R}^n, L^p(x_0,\xi)$ and $L^p(x_0,-\xi) = (-1)^m L^p(x_0,\xi)$, they must have the same sign. This implies that m, it is an even integer

Definition for Elliptic in Domain:

Let $\Omega \in \mathbb{R}^n$, it is a domain and we say that a linear partial differential operator

$$L(x,D) = \sum_{|\alpha|=2k} a_\alpha(x)\xi^\alpha 0, \forall \; x \in \Omega, \xi \in \mathbb{R}^n/\{0\} \longrightarrow \quad (1.15)$$

And also L, it is uniformly Elliptic in Ω and if there exists a constant $\Theta > 0$, such that

$$L(x,D) = (-1)^k \sum_{|\alpha|=2k} a_\alpha(x)\xi^\alpha \geq \Theta|\xi|^{2k}, \forall \; x \in \Omega, \xi \in \mathbb{R}^n/\{0\} \longrightarrow \quad (1.16)$$

Definition for Divergence: [2], [15]

An operator is divergence form if there are functions $a_{\sigma\gamma}: \Omega \longrightarrow \mathbb{R}$, and then

$$L(x,D)u = \sum_{0\leq|\sigma|,|\gamma|\leq k} (-1)^{|\sigma|} D^\sigma\left(a_{\sigma\gamma}(x)D^\gamma u\right) \longrightarrow \quad (1.17)$$

Remark:

An operator in divergence form is Elliptic if and only of

$$\sum_{|\sigma|,|\gamma|=k} \xi^\sigma\left(a_{\sigma\gamma}(x)\xi^\gamma\right) > 0, \forall x \in \Omega, \xi \in \mathbb{R}^n/\{0\} \longrightarrow (1.18a)$$

And also the operator is uniformly Elliptic if and only if there exists a constant $\Theta > 0$, such that

$$\sum_{|\sigma|,|\gamma|=k} \xi^\sigma\left(a_{\sigma\gamma}(x)\xi^\gamma\right) > \Theta|\xi|^{2k}, \forall x \in \Omega, \xi \in \mathbb{R}^n/\{0\} \longrightarrow \quad (1.18b)$$

Lemma – 3:

Let

$a_\alpha \in C_b^{|\alpha|-k}(\overline{\Omega}),\; for\; k < |\alpha| < 2k$, and $\longrightarrow$ (1.19)

$a_\alpha \in C_b(\overline{\Omega}),\; for\; |\alpha| \leq k \longrightarrow$ (1.20)

And then there exists $a_{\sigma\gamma} \in C_b^{|\sigma|}(\overline{\Omega})$, such that for every $u \in C^{2k}(\Omega)$, we have

$$L(x,D)u = \sum_{|\alpha|\leq 2k} a_\alpha(x)D^\alpha u \longrightarrow \quad (1.21)$$

And $L(x,D)u = \sum_{|\alpha|\leq 2k}(-1)^{|\sigma|}D^{|\alpha|}\left(a_{\sigma\gamma\gamma}(x)D^\gamma u\right) \longrightarrow$ (1.22)

Proof:

For every $|\alpha| \leq 2k$, choose σ_α and γ_α satisfying $|\sigma_\alpha|, |\gamma_\alpha| \leq k$ $and\; also$ $\sigma_\alpha + \gamma_\alpha = \alpha$

Note that this choice of choosing is not unique, and now for any $u \in C^{2k}(\Omega)$ and $\emptyset \in D(\Omega)$, we get

$$\int_\Omega L(x,D)u\emptyset dx = \sum_{|\alpha|\leq 2k} \int_\Omega (D^\alpha u)a_\alpha \emptyset dx = \sum_{|\alpha|\leq 2k} \int_\Omega (D^{\sigma_\alpha+\gamma_\alpha}u)a_{(\sigma_\alpha+\gamma_\alpha)} \emptyset dx$$

$\Rightarrow \int_\Omega L(x,D)u\emptyset dx = \sum_{|\alpha|\leq 2k} \int_\Omega D^{\sigma_\alpha}(D^{\gamma_\alpha}u)a_{(\sigma_\alpha+\gamma_\alpha)} \emptyset dx$, using integration by parts we get

$$\Rightarrow \int_\Omega L(x,D)u\emptyset dx = \sum_{|\alpha|\leq 2k}(-1)^{|\sigma_\alpha|} \int_\Omega (D^{\gamma_\alpha}u)D^{\sigma_\alpha}\left[a_{(\sigma_\alpha+\gamma_\alpha)}\emptyset\right] dx$$

$$\Rightarrow \int_\Omega L(x,D)u\emptyset dx = \sum_{|\alpha|\leq 2k}(-1)^{|\sigma_\alpha|} \int_\Omega (D^{\gamma_\alpha}u) \sum_{\rho\leq\sigma_\alpha} \binom{\sigma_\alpha}{\rho} D^{\sigma_\alpha-\rho}\left[a_{(\sigma_\alpha+\gamma_\alpha)}\right]D^\rho\emptyset\, dx$$

$\Rightarrow \int_{\Omega}^{\cdot} L(x,D)u\emptyset dx =$

$\sum_{|\alpha|\leq 2k,\rho\leq\sigma_\alpha}(-1)^{|\sigma_\alpha|+|\rho|}\int_{\Omega}^{\cdot}(D^\rho)\binom{\sigma_\alpha}{\rho}\sum_{\rho\leq\sigma_\alpha}\left[D^{\sigma_\alpha-\rho}a_{(\sigma_\alpha+\gamma_\alpha)}D^{\gamma_\alpha}u\right]\emptyset\, dx$

$\Rightarrow \int_{\Omega}^{\cdot} L(x,D)u\emptyset dx = \sum_{0\leq|\sigma|,|\gamma|\leq k}(-1)^{|\sigma|}D^\sigma\left[a_{\sigma\gamma}(x)D^\gamma u\right]\emptyset dx$

Note that the last equality is a definition holds $\forall\ \emptyset \in D(\Omega)$, and hence completes the proof

1. Existence and Uniqueness of solution of Dirichlet Problem
2. The Dirichlet Problem – Types of Solution

Definition for Classical Solution: [7], [6], [9]

Let $\Omega \subset \mathbb{R}^n$, it is a bounded domain and suppose $f \in C_b(\Omega)$, it is given. A function $u \in C_b^{2k}(\Omega) \cap F_b^{2k-1}(\overline{\Omega})$, it is a **classical solution** of the Dirichlet problem [2], [3], [4], [15]

If $L(x,D)u = \sum_{0\leq|\sigma|,|\gamma|\leq k}(-1)^{|\sigma|}D^\sigma\left[a_{\sigma\gamma}(x)D^\gamma u\right] = f\ \ in\ \ \Omega$ and ⟶ (2.1)

$D^\alpha u = 0\ \ for\ |\alpha| \leq k-1\ \ in\ \partial$ ⟶ (2.2)

One of the most important ideas of the Modern Analysis is that we want to guarantee the existence of solution to a problem; it is usually easier to do so in a "Bigger" space of functions. This is clearly the case with the classical Dirichlet problem, although we might expect a solution to hall all of the smoothness suggested at first. The first step in relaxing the conditions on the solution is to state the problem in terms of Sobolev Spaces.

Definition for Strong Solution:

Let $\Omega \subset \mathbb{R}^n$, it is a bounded domain and suppose $f \in L^2(\Omega)$, it is given. A function $u \in H^k(\Omega) \cap H_0^k(\overline{\Omega})$, it is a **strong solution** of the Dirichlet problem, if

$L(x,D)u = \sum_{0\leq|\sigma|,|\gamma|\leq k}(-1)^{|\sigma|}D^\sigma\left[a_{\sigma\gamma}(x)D^\gamma u\right] = f\ \ in\ \ \Omega$ ⟶ (2.3)

Notes:

1. We have relaxed the conditions not only on the solution u,but also on the data f, and the space $L^2(\Omega)$, it is certainly the obvious space for f, once we have relaxed the conditions on u, so the additional generality will come along "For Free" (In fact, we will be able to weaken the conditions on f, each time we relax the conditions on the solution, as we shall see below)
2. For classical solution, the differential equation (D.E. 2.1) is taken to hold in a point – wise sense. For strong solutions, (D.E. 2.3) is understood either in terms of equivalence classes (The right and left sides of the equation represent the same equivalence class of sequence in the $L^2(\Omega)$ norm) or in a "almost everywhere (a. e) sense"

3. Instead of imposing boundary conditions explicitly as we did in the classical problem, we have incorporated then into the $H_0^k(\overline{\Omega})$, in the new problem
4. By combining the previous observations we see that the new problem is indeed a generalization of the classical problem, i.e. any classical solution of the Dirichlet problem is also is a strong solution.

We now take further steps in weakening the condition on solution of the Dirichlet problem: We state the problem in variational form. The first step is create a bilinear form from the differential operator Lusing integration by part

Let us take $L(x,D)u = \sum_{0\le|\sigma|,|\gamma|\le k}(-1)^{|\sigma|}D^{\sigma}\left[a_{\sigma\gamma}(x)D^{\gamma}u\right]$, and then

$$\int_{\Omega} \emptyset Lu\,dx = \sum_{0\le|\sigma|,|\gamma|\le k}(-1)^{|\sigma|}\int_{\Omega} \emptyset D^{\sigma}\left[a_{\sigma\gamma}(x)D^{\gamma}u\right]dx \longrightarrow \quad (2.4)$$

By using integration by part, we get

$$\int_{\Omega} \emptyset Lu\,dx = \sum_{0\le|\sigma|,|\gamma|\le k}(-1)^{|\sigma|}(-1)^{|\gamma|}\int_{\Omega}(D^{\sigma}\emptyset)\left[a_{\sigma\gamma}(x)D^{\gamma}u\right]dx$$

$$\Longrightarrow \int_{\Omega} \emptyset Lu\,dx = \sum_{0\le|\sigma|,|\gamma|\le k}(-1)^{|\sigma|+|\gamma|}\int_{\Omega}(D^{\sigma}\emptyset)\left[a_{\sigma\gamma}(x)D^{\gamma}u\right]dx$$

$$\Longrightarrow \int_{\Omega} \emptyset Lu\,dx = \sum_{0\le|\sigma|,|\gamma|\le k}(-1)^{2|\sigma|}\int_{\Omega}(D^{\sigma}\emptyset)\left[a_{\sigma\gamma}(x)D^{\gamma}u\right] =$$

$$\sum_{0\le|\sigma|,|\gamma|\le k}\int_{\Omega}\left[a_{\sigma\gamma}(x)D^{\gamma}u\right](D^{\sigma}\emptyset)dx$$

Define $B[v,u] = \int_{\Omega}\left[a_{\sigma\gamma}(x)D^{\gamma}u\right](D^{\sigma}v)dx$

To be bilinear form associated with the Elliptic Partial Differential Operator (EPDO) L, and also $B[v,u]$, it is well defined for u *and* v that are merely in $H^k(\Omega)$

Definition for Weak Solution:

Let $\Omega \subset \mathbb{R}^n$ it is a bounded domain and suppose suppose $f \in H^{-k}(\Omega)$, it is given. A function

$u \in H_0^k(\Omega)$, it is a **weak solution** of the Dirichlet problem, if

$$B[v,u] = f(v) \quad v \in H_0^k(\Omega) \longrightarrow \quad (2.5)$$

And also by (2.4) (For $v \in H_0^k(\Omega)$ in place of $\emptyset \in D(\Omega)$), we conclude that any strong solution of the Dirichlet problem is automatically a weak solution. However, since we require so much less smoothness of weak solutions than strong ones, it will be far easier to show that if Ω, f, and the coefficients $a_{\sigma\gamma}$, they are sufficiently "Nice" the weak solution is in fact, a strong solution or a classical solution.

The Lax – Milgram Lemma: [10], [13]

Let H, it is a Hilbert space and let $B\colon H \times H \longrightarrow \mathbb{R}$, it is a bilinear mapping and supposes that there exists a positive constants c_1 *and* c_2 such that

$|B[x,y]| \leq c_1\|x\|_H\|y\|_H, \forall\ x,y \in H \longrightarrow$ (2.6)

And $B[x,x] \geq c_2\|x\|_H^2\ \ \forall\ x \in H \longrightarrow$ (2.7)

Then, $\forall\ f \in H^*$ and there exists a unique $y \in H$ such that

$B[x,y] = f(x),\ \forall\ x \in H \longrightarrow$ (2.8)

Furthermore, there exists a constant C , independent of f, such that $\|y\|_H \leq C\|f\|_{H^*}$

Note:

A mapping B, satisfying *2.7) for some $c_2 > 0$, it is called **coercive**

We now prove the basic energy or coercivity estimate for the elliptic Dirichlet problem.

Lemma – 5:

Let $f,g \in S(\mathbb{R}^n)$, and then $\langle f,g\rangle = \langle \hat{f},\hat{g}\rangle$

Proof:

Since $\langle f,g\rangle = \int_{\mathbb{R}^n}^{\cdot} \overline{f(x)}\, g(x)dx = \int_{\mathbb{R}^n}^{\cdot} \overline{f(x)} \int_{\mathbb{R}^n}^{\cdot} \widehat{g(\xi)} e^{i\xi x}\, dx d\xi =$

$\int_{\mathbb{R}^n}^{\cdot} \hat{g}(\xi) \int_{\mathbb{R}^n}^{\cdot} \overline{f(x)e^{-i\xi x}}\, dx\, d\xi \implies \langle f,g\rangle = \int_{\mathbb{R}^n}^{\cdot} \overline{\widehat{f(x)}}\, \widehat{g(\xi)} d\xi = \langle \hat{f},\hat{g}\rangle$, and hence completes the proof

Lemma – 6:

$L^2(\mathbb{R}^n)$, it is a Hilbert space with inner product $\langle f,g\rangle = \int_{\Omega}^{\cdot} \overline{f} g$

Proof:

By the lemma – 5 it is obviously true

Garding's Inequality: [4], [5], [10]

Let Ω it is a bounded domain with the K extension property. Let $L(x,D)$, it is linear partial differential operator in divergence form of order $2k$ such that for some **ϴ**>0, and then the uniform ellipticity condition holds. Also supposes that

$a_{\sigma\gamma} \in C_b(\overline{\Omega}), \forall\ |\sigma| = |\gamma| = k$, and $\longrightarrow$ (2.9)

$a_{\sigma\gamma} \in L^\infty(\Omega), \forall\ |\sigma|,|\gamma| \leq k \longrightarrow$ (2.10)

Then there exists constants $c_3\ \ and\ \lambda_G \geq 0$, such that

$B[u,u] + \lambda_G\|u\|_{L^2(\Omega)}^2 \geq c_3\|u\|_{H^k(\Omega)}^2, \forall\ u \in H_0^k(\Omega) \longrightarrow$ (2.11)

This inequality we can prove easily by using Holder's inequality (For the complete proof see [9])

Remarks:

In the proof of Garding's Inequality: we have the following results

1. When $u \in H_0^k(\Omega)$, the results of Garding's inequality $B[u,u]$, it is splitting into principal part and lower order terms, i.e. $B[x,x] = I_1 + I_2$, where
   $I_1 = \sum_{|\sigma|=|\gamma|=k} \int_\Omega^{\cdot} [a_{\sigma\gamma}(x) D^\gamma u](D^\sigma u)dx$; $I_2 =$
   $\sum_{0\le|\sigma|,|\gamma|\le k, |\sigma|+|\gamma|\le 2k} \int_\Omega^{\cdot} [a_{\sigma\gamma}(x) D^\gamma u](D^\sigma u)dx$
2. Let $B(x_0, \delta)$, for some $x_0 > 0$ and sufficiently small and then I_1 $when\, u \in H_0^k(\Omega)$, we have
   $I_1 = I_{11} + I_{12}$, where
   $I_{11} = \sum_{|\sigma|=|\gamma|=k} \int_{\mathbb{R}^n}^{\cdot} [a_{\sigma\gamma}(x_0) D^\gamma u](D^\sigma u)dx$, and also
   $I_{12} = \sum_{|\sigma|=|\gamma|=k} \int_{\mathbb{R}^n}^{\cdot} [a_{\sigma\gamma}(x) a_{\sigma\gamma}(x_0) - D^\gamma u](D^\sigma u)dx$
3. Continue with the estimate of I_1, in the general case $when\, u \in H_0^k(\Omega)$, we use the partition of unity, so that by covering $\overline{\Omega}$ with the finite collection of balls:
   $B_i = B(x, \delta_i)\ for\ i = 1, 2, \dots, m\ with\ x_i \in \Omega and\ \delta_i > 0$

From the remarks, we conclude that when ψ_i, it is the partition of unity on $\overline{\Omega}$ subordinate to the covering B_i and then the set $\phi_i(x) = \left(\frac{\phi_i^2}{\sum_{j=1}^m \psi_j^2}\right)^{\frac{1}{2}}$, in this situation we have

a. $0 \le \phi_i(x) \le 1$
b. $\phi_i \in C^\infty(B_i \cap \Omega)$
c. $\sum_{i=1}^m \phi_i^2 = 1\ \forall\ x \in \Omega$, and
d. $u_i = u\phi_i \in H_0^k(B_i)$

Garding's inequality is easily verified for second order PDE [14] i.e. in this case where $L(x,D)$, it is the second order differential operator of the form

$$L(x,D)u = \sum_{i,j}^n \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \sum_{i=1}^n \frac{\partial}{\partial x_i} b_i(x) + c(x)u \longrightarrow \quad (2.12)$$

With the corresponding bilinear form

$$B[v,u] = -\sum_{i,j}^n \int_\Omega^{\cdot} a_{ij}(x) u_{x_i} v_{x_j} dx + \int_\Omega^{\cdot} \sum_{i=1}^n b_i(x) u_{x_i} v dx + \int_\Omega^{\cdot} c(x) uv dx \longrightarrow (2.13)$$

In this case we do not need to use either Fourier transforms or the partition of unity technique, and the proof can be carried out under weaker hypotheses on the higher order coefficients.

Theorem:

Let Ω it is a bounded domain and let $L(x,D)$, it is the second order differential operator in divergence form of the formed described in (2.12) such that for **ϴ**>0, the uniform ellipticity condition holds. Also supposes that $a_{ij}, b_k \in L^{\infty}(\Omega), for\ i,j = 1, \dots, n\ and\ k = 0, \dots, n$, and then there exists constants $c_3\ and\ \tau_G \geq 0$, such that

$B[v,u] + \tau_G \|u\|^2_{L^2(\Omega)} \geq c_3 \|u\|^2_{H^1(\Omega)}\ where\ B$, it is defined in (2.13)

Proof:

By using uniform ellipticity condition and Holder's inequality we get

$B[u,u] = -\sum_{i,j}^{n} \int_{\Omega} a_{ij}(x)\, u_{x_i} u_{x_j} dx + \int_{\Omega} \sum_{i=1}^{n} b_i(x)\, u_{x_i} u dx + \int_{\Omega} c(x)\, u^2 dx$

To be bilinear form associated with elliptic partial differential operator L, as in (2.13) we have

$B[u,u] \geq \mathbf{\Theta} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \sum_{i=1}^{n} b_i(x)\, u_{x_i} u dx + \int_{\Omega} c(x)\, u^2 dx$

$\Rightarrow B[u,u] \geq \mathbf{\Theta} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |b_i(x) u_{x_i} u|\, dx - \int_{\Omega} |c(x) u^2|\, dx$

$\Rightarrow B[u,u] \geq \mathbf{\Theta} \int_{\Omega} |\nabla u|^2 dx - \sum_{i=1}^{n} \|b_i(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} |u_{x_i} u|\, dx - \|c(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} u^2\, dx$

$\Rightarrow B[u,u] \geq$

$\int_{\Omega} |\nabla u|^2 dx - \|b_1(x)\|_{L^{\infty}(\Omega)} \left(\int_{\Omega} |u_{x_1} u|\, dx + \cdots + \|b_n(x)\|_{L^{\infty}(\Omega)} \right) - \int_{\Omega} |u_{x_n} u|\, dx -$

$\|c(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} |u|^2\, dx$

Choose $\max \|b_i(x)\|_{L^{\infty}(\Omega)}$, we get

$B[u,u] \geq \mathbf{\Theta} \int_{\Omega} |\nabla u|^2 dx - \max \|b_i(x)\|_{L^{\infty}(\Omega)} \sum_{i=1}^{n} \int_{\Omega} |u_{x_i} u|\, dx - \|c(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} |u|^2\, dx$

$\Rightarrow B[u,u] \geq \Theta \int_{\Omega} |\nabla u|^2 dx - \max \|b_i(x)\|_{L^{\infty}(\Omega)} \sum_{i=1}^{n} \int_{\Omega} |u_{x_i}||u|\, dx - \|c(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} |u|^2\, dx$

$\Rightarrow B[u,u] \geq \Theta \int_{\Omega} |\nabla u|^2 dx - \max \|b_i(x)\|_{L^{\infty}(\Omega)} \sum_{i=1}^{n} \int_{\Omega} |\nabla u||u|\, dx - \|c(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} |u|^2\, dx$

Use (2.12), and Poincare's inequality, we get

$\Rightarrow B[u,u] \geq$

$\Theta \int_{\Omega} |\nabla u|^2 dx - \max \|b_i(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} \left[\varepsilon |\nabla u|^2 + \frac{1}{4\varepsilon} |u|^2 \right] dx - \|c(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} |u|^2\, dx$

$\Rightarrow B[u,u] \geq \Theta \int_{\Omega} |\nabla u|^2 dx - \varepsilon \max \|b_i(x)\|_{L^{\infty}(\Omega)} \int_{\Omega} \left[|\nabla u|^2 - \frac{\max \|b_i(x)\|_{L^{\infty}(\Omega)}}{4\varepsilon} \int_{\Omega} |u|^2 dx \right] dx -$

cxL∞ΩΩ.u2dx

Take $\tau_G = \frac{\max \|b_i(x)\|_{L^{\infty}(\Omega)}}{4\varepsilon}\ and\ \frac{\Theta}{2} = \varepsilon \max \|b_i(x)\|_{L^{\infty}(\Omega)}$, substitute in as above inequality we get

$\Rightarrow B[u,u] \geq \left[\Theta - \frac{\Theta}{2} \right] \int_{\Omega} [|\nabla u|^2 dx - (c_3 + c_2)|u|^2] \int_{\Omega} |u|^2\, dx$ (By lax Milgram inequality and given c_3)

Since $\left[\Theta - \frac{\Theta}{2} \right] \int_{\Omega} [|\nabla u|^2 dx - (c_3 + c_2)|u|^2] \int_{\Omega} |u|^2\, dx = \frac{\Theta}{2} \int_{\Omega} |\nabla u|^2 dx - \tau_G \|u\|_2^2$

$\Rightarrow B[u,u] \geq \frac{\Theta}{2} \int_{\Omega} |\nabla u|^2 dx - \tau_G \|u\|_2^2 = \frac{\Theta}{2} \|\nabla u\|_2^2 - \tau_G \|u\|_2^2$

Again use Poincare's inequality we get

$\Longrightarrow B[u,u] \geq c_3\|u\|_{1,2}^2 - \tau_G\|u\|_2^2$, and hence completes the proof

2.4 Existence of Weak Solutions

In this stage, we ready to prove our basic existence result for weak solutions

Theorem:

Let $L(x,D)$, it is the linear partial differential operator in divergence form of order $2k$, satisfying the hypothesis of Garding's Inequality and then there exists $\tau_G \geq 0$ such that, for any

$\tilde{\tau} \geq \tau_G$ *and* $f \in H^{-k}(\Omega)$, the Dirichlet problem for the operator

$\tilde{L}(x,D) = L(x,D) + \tilde{\tau}$ ⟶ (2.14)

It has a unique weak solution $u \in H_0^k(\Omega)$. Furthermore, this solution satisfies

$\|u\|_{k,2} \leq C\|f\|_{-k,2}$ ⟶ (2.15)

Proof:

By Garding's Inequality, there exists $\tau_G \geq 0$, such that the elementary inequality

$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ *where* $a,b > 0$ *and* ε , it is too small

Since $\tilde{\tau} \geq \tau_G$, and also $\tilde{B}[u,v] = B[u,v] + \tilde{\tau}(u,v)_{L^2(\Omega)}$ ⟶ (2.16)

Equation (2.16) is a bilinear form associated with the operator $\tilde{L}$

To prove that $\tilde{B}$, satisfies the hypotheses of Lax – Milgram lemma

Let $H = H_0^1(\Omega)$ *and* $u,v \in H$, and then

$\left|\tilde{B}[u,v]\right| = |B[u,u] + \tilde{\tau}(u,v)| \leq |B[u,v]| + |\tilde{\tau}|(u,v)$

Since$|B[u,v]| + |\tilde{\tau}|(u,v) = \left|\sum_{0\leq|\sigma|,|\gamma|\leq k}\int_\Omega^{\cdot}\left[a_{\sigma\gamma}(x)D^\gamma u D^\sigma v\right]dx\right| + |\tilde{\tau}|(u,v)$

$\Longrightarrow |B[u,v]| + |\tilde{\tau}|(u,v) \leq \sum_{0\leq|\sigma|,|\gamma|\leq k}\int_\Omega^{\cdot}\left|a_{\sigma\gamma}(x)\right||D^\gamma u||D^\sigma v|dx + |\tilde{\tau}|(u,v)$

$\Longrightarrow |B[u,v]| + |\tilde{\tau}|(u,v) \leq \sum_{0\leq|\sigma|,|\gamma|\leq k}\int_\Omega^{\cdot}\max_{0\leq|\sigma|,|\gamma|\leq k}\sup\left|a_{\sigma\gamma}(x)\right||D^\gamma u||D^\sigma v|dx +$ $|\tilde{\tau}|(u,v)$

$\Longrightarrow |B[u,v]| + |\tilde{\tau}|(u,v) \leq \max_{0\leq|\sigma|,|\gamma|\leq k}\left\|a_{\sigma\gamma}\right\|_{L^\infty(\Omega)}\sum_{0\leq|\sigma|,|\gamma|\leq k}\int_\Omega^{\cdot}|D^\gamma u||D^\sigma v|dx + |\tilde{\tau}|(u,v)$

$\Longrightarrow |B[u,v]| + |\tilde{\tau}|(u,v) \leq C\|v\|_H\|u\|_H$

$\Longrightarrow \tilde{B}[u,v] \geq c_2\|x\|_H^2,\ \forall\ x \in H$

Again by Garding's inequality, we have

$\tilde{B}[u,u] = \tilde{\tau}\|u\|_2 + B[u,v] \geq c_3\|u\|_k^2$

Thus $\tilde{B}$ satisfies $B[x,y] = f(x), \forall\ x \in H$, i.e. $B[x,x] \geq c_2\|x\|_k^2, \forall\ x \in H$

Thus, Lax – Milgram lemma guarantee that $\forall\ f \in H^{-k} = H^{*}$, and then there exists a unique weak solution $u \in H$ of the Dirichlet problem, and that the solution satisfies the estimate $\|u\|_{k,2} \leq C\|f\|_{-k,2}$

Hence completes the proof.

CONCLUSION

In this research article we conclude that existence and uniqueness solution for a class of linear elliptic equations of Dirichlet problem in divergence form will be obtained in weak solution condition.

FUTURE IMPLEMENTS

From this article any one will do non linear uniqueness of weak and also strong solution for elliptic equations of higher order in divergence form of the Dirichlet problems

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