SPACE OF MULTIPLIERS AS A DUAL SPACE

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Let E be a homogeneous Banach space of distributions and $m(E,E)$ be the space of multipliers from E to E. It is shown that $m(E,E)$ is isometrically isomorphic to the dual space of a certain Banach space of continuous functions.

1. INTRODUCTION

In [4] and [5] some results of Fourier Analysis, known for L^p, C(the space of continuous functions), M(the space of measures) and Orlicz spaces etc., were studied for Banach spaces of distributions (briefly called BD-spaces). In [2], spaces A_p($1 < p < \infty$) were defined and it was proved that $M_p(m(L^p, L^p)$ in our notation) is isometrically isomorphic to the dual space of A_p. In this paper, we aim to extend this result to homogeneous BD-spaces. In section 3, we define the space $A(E,E)$ and prove that if E is a homogeneous BD-space, then $A(E,E)$ is also a homogeneous BD-space. Next, we show that the space of multipliers $m(E,E)$ is isometrically isomorphic to the dual space of $A(E,E)$.

2. DEFINITIONS AND NOTATIONS

We refer to [1] for all the standard definitions, notations and assumptions. By G, we shall denote the circle group $R/2\pi Z$. All our functions and distributions are assumed to be defined on the circle group G. By D, we shall denote the space of all distributions on G.

A Banach space E will be called a Banach space of distributions or a BD-space, if it is continuously embedded in D; and regarded as a subset of D, it satisfies the following properties:

(2.1) $C^\infty \subset E$; (2.2) $f \in E \Rightarrow T_x f \in E$ and $\| T_x f \|_E = \| f \|_E$ for all $x \in G$; (2.3) $f \in E \Rightarrow f^\vee \in E$ and $\| f^\vee \|_E = \| f \|_E$

Where $f^\vee(u) = f(u^\vee)$ for each $u \in C^\infty$ and $u^\vee(x) = u(-x)$ for all $x \in G$.

The spaces L^p ($1 \le p \le \infty$), C, M and Orlicz spaces etc. are all BD-spaces. Throughout this paper, E will always denote a BD-space.

A BD-space E is said to be homogeneous if for every $f \in E$, the function $x \to T_x f$ is continuous from G to E.

Let FE denote the set of transforms $\hat{f}$ of elements f of a BD-space E. Then by (E,E) we denote the set of all complex valued functions ϕ on Z such that

$f \in E \Rightarrow \phi \cdot \hat{f} \in FE$.

Such functions ϕ are called multipliers of type (E, E).

To each $\phi \in (E, E)$, there corresponds a multiplier operator U_ϕ(see [1, 16.2]) from E to E, defined by the relation

$$(U_\phi f)^\wedge = \phi \cdot \hat{f}, \qquad \forall f \in E.$$

By an m-operator of type (E, E), we shall mean a linear operator U from E to E such that

$$U(T_x * f) = t * Uf$$

for each trigonometric polynomial t and each $f \ in\ E$. By $m(E, E)$ we shall denote the set of all m-operators of type (E, E). If $U \in m(E, E)$, then its norm is denoted by $\| U \|_{m(E,E)}$. It can easily be proved that $U \in m(E, E)$ iff $U = U_\phi$ for some $\phi \in (E, E)$(see [1, 16.2]).

3. THE SPACE $A(E, E)$

In this section, we define and discuss the space $A(E, E)$ and prove that $m(E, E)$ is isometrically isomorphic to the dual space of $A(E, E)$.

3.1. Definition

Let E be a homogeneous BD-space. The space $A(E, E)$ is defined as the subspace of C consisting of those functions h which can be represented as

$h = \sum_{i=1}^{\infty} f_i * g_i,$ where $f_i \in E$, $g_i \in E^*$.

$$\sum_{i=1}^{\infty} \| f_i \|_E \| g_i \|_{E^*} < \infty.$$

Note that if E is a homogeneous BD-space, then E^* is also a BD-space and $f_i * g_i \in C$ by Theorem 1 and Theorem 3 of [4]. The norm of h in $A(E, E)$ is taken as

$$\| h \|_{A(E,E)} = inf \sum_{i=1}^{\infty} \| f_i \|_E \| g_i \|_{E^*},$$

where the infimum is taken over all the representations of h.

3.2. Lemma

Let E be a homogeneous BD-space and $U \in m(E, E)$. Then there exists a sequence $\{h_n\}_{i=1}^{\infty}$ in C^∞ such that, for each $f \in E$,

$$lim_{n\to\infty} h_n * f = Uf and\ for\ each\ n, \| h_n * f \|_E \le \| U \|_{m(E,E)} \| f \|_E.$$

Proof. Let $U \in m(E, E)$ and let $\{h_n\}_{i=1}^{\infty}$ be an approximate identity composed of trigonometric polynomials k_n such that $\| k_n \|_1 \le 1\ define\ h_n = Uk_n$,

$n = 1,2,3, \ldots, which\ are\ also\ trigonometric\ polynomials.$ Then, for $f \in E$,

$\| Uk_n * f \|_E \leq \| k_n * Uf \|_E \leq \| k_n \|_1 \| Uf \|_E \leq \| U \|_{(E,E)} \| f \|_E$.

for every n. Further

$h_n * f = Uk_n * f = k_n * Uf \to Uf$

in $E\ as\ n \to \infty$.

3.3. Theorem

Let E be a homogeneous BD-space. Then $A(E,E)$ is also a homogeneous BD-space.

Proof. $A(E,E)$ is clearly a normed linear space. To prove the completeness, suppose that $\{h_n\}_{i=1}^{\infty}$ is a Cauchy sequence in $A(E,E)$. Now, choose a subsequence $\{u_n\}$ of $\{h_n\}$ such that

$$\| u_{n+1} - u_n \|_{A(E,E)} < \frac{1}{2^n}, \quad n = 1,2,3, \ldots$$

From the definition of $A(E,E)$, we can find $\{f_{n,j}\}\ in\ E$ and $\{g_{n,j}\}\ in\ E^*$ ($j = 1,2,3, \ldots$) such that

$$(i) \quad u_1 = \sum_i f_{1,j} * g_{1,j};\ (ii) \quad \sum_j \| f_{1,j} \|_E \| g_{1,j} \|_{E^*} < \| u_1 \|_{A(E,E)} + 1 = C + 1\ (say),$$

$$u_{n+1} - u_n = \textstyle\sum_i f_{n+1,j} * g_{n+1,j}, \qquad (iii)$$

$$\textstyle\sum_j \| f_{n+1,j} \|_E \| g_{n+1,j} \|_{E^*} < \frac{1}{2^n}, \quad n = 1,2,3, \ldots$$

Set $u = u_1 + \sum (u_{n+1} - u_n).$ Then $u \in A(E,E)$, since

$$(iv) \quad \sum \| f_{1,j} \|_E \| g_{1,j} \|_{E^*} + \sum_n \sum_j \| f_{n+1,j} \|_E \| g_{n+1,j} \|_{E^*} \leq C + 2.$$

Let $\varepsilon > 0$ and N be such that $2^{-N} < \varepsilon$. Then, for $n > N$,

$$\| u - u_{n+1} \|_{A(E,E)} = \left\| u - \left[u_1 + \sum_{r=1}^{n} (u_{r+1} - u_r)\right] \right\|_{A(E,E)} \leq \sum_{r=n+1}^{\infty} \left[\sum \| f_{r+1,j} \|_E \| g_{r+1,j} \|_{E^*} \right]$$

$$\leq \sum_{j=n}^{\infty} 2^{-r} < \varepsilon.$$

Therefore, $\{u_n\}$ converges to u in $A(E,E)$. Hence $A(E,E)$ is a Banach space.

$A(E,E)$ is continuously embedded into C because $\| h_n \|_\infty \leq \| h \|_{A(E,E)}\ for\ every\ h \in A(E,E)$ since C is continuously embedded into D, $A(E,E)\ is\ countinuously\ embedded$ in D. Furthermore, since $C^\infty \subset E, C^\infty \subset E^*$.

Now, let $h \in A(E,E)$ and let $\sum_{i=1}^{\infty} f_i * g_i$, be a representation of h. Then, for each $x \in G$,

$T_x h = \sum_{i=1}^{\infty} f_i * g_i$. Since $T_x f_i \in E$ and $\| T_x f_i \|_E = \| f_i \|_E\ for\ every\ x \in G,$ we obtain that, for every $x \in G,$

$$\sum_i \| T_x f_i \|_E \| g_i \|_{E^*} = \sum_i \| f_i \|_E \| g_i \|_{E^*} < \infty.$$

Hence, for each $x \in G$, $T_x h \in A(E,E)$ and $\| T_x h \|_{A(E,E)} = \| h \|_{A(E,E)}$. Making use of (2.3) for E and E^* both, we obtain without much difficulty that $h^\vee \in A(E,E)$ and $\| h^\vee \|_{A(E,E)} = \| h \|_{A(E,E)}$. Homogeneity of $A(E,E)$ follows from the homogeneity of E, since $\| T_x f_i - f_i \|_E \to 0\ for\ each\ i,$ implies that

$\| T_x h - h \|_{A(E,E)} \leq \sum \| T_x f_i - f_i \|_E \| g_i \|_{E^*} \to 0.$

Thus, $A(E,E)$ is a homogeneous Banach space of distributions.

3.4. Theorem Let E be a homogeneous BD-space. Then $m(E,E)$ is isometrically isomorphic to the dual space of $A(E,E)$. The span of translation operators on E is weak dense in $m(E,E)$.

Proof. Suppose $U \in m(E,E)$. Define the linear functional T on $A(E,E)$ by

$$T(h) = \sum_{i=1}^{\infty} U f_i * g_i(0),$$

where $\sum f_i * g_i$ is a representation of h as an element of $A(E,E)$. T is well defined, i.e., $T(h)$ is independent of the particular representation of h chosen. To see this, suppose that $\sum_{i=1}^{\infty} f_i * g_i = 0$ is a representation of 0 as an element of $A(E,E)$. Choose a sequence $\{h_n\}_{n=1}^{\infty}$ satisfying the conclusions of Lemma 3.2. Then,

$$\sum_{i=1}^{\infty} U f_i * g_i(0), = lim_{n\to\infty} \sum_{i=1}^{\infty} h_n * f_i * g_i(0),$$

Since the series $\sum_{i=1}^{\infty} h_n * f_i * g_i(0),$ is convergent uniformly over the set of natural numbers and, for each i , $h_n * f_i \to U f_i$ in E as n $\to \infty$ Again, $h_n \in C^\infty$ and $\sum_{i=1}^{\infty} f_i * g_i$ Converges in C.

Hence,

$$\sum_{i=1}^{\infty} h_n * f_i * g_i(0), = \sum_{i} \int [h_n(-x)][(f_i * g_i)(x)]\,dx = \int [h_n(-x)][\sum f_i * g_i(x)]dx = 0,$$

$= 0$

Thus, T is well defined. Further,

$$| T(h) | \leq \sum_{i} | Uf_i * g_i(0) | \leq \Sigma \sum_{i} \| Uf_i \|_E \| g_i \|_{E^*} \leq \| U \|_{m(E,E)} \Sigma \| f_i \|_E \| g_i \|_{E^*}.$$

The above inequality holds for all representations of h. Hence,

$| T(h) | \leq \| U \|_{(E,E)} \| h \|_{A(E,E)}.$

This implies that T is continuous on $A(E,E)$ and

(3.1) $\| T \| \leq \| U \|_{(E,E)}.$

Further, (see [3, 4.4])

(3.2) $\| U \|_{(E,E)} \leq sup\ sup\ \{: \| f \|_E \leq 1, \| g \|_{E^*} \leq 1\}$

$\leq sup\ sup\ \{: \| h \|_{A(E,E)} \leq 1\} = \| T \|.$

From (3.1) and (3.2),

$\| T \| = \| U \|_{(E,E)}.$

In order to show that $U \to T$ is onto, suppose that $T \in A(E,E)^*$. For $g \in E^*$, define the linear functional $f \to T(f * g)$ on E. Since $T \in A(E,E)^*$.

(3.3) $| T(f * g) | \leq \| T \| \| f \|_E \| g \|_{E^*}.$

Since E^* is the dual space of E, for each $g \in E^*,$ there exists a unique element $of\ E^*,$ say Ug, with Ug(f) = $Ug * f(0) = T(f * g).$ Now, for $g \in E^*$ and any trigonometric polynomial t,

$U(t * g) * f(0) = T(t * f * g)$

$= U_g(t * f)(0)$

$= (U_g * t) * f(0)$

Hence, $U(t * g) = t * Ug,$ where g and t are as above. This shows that $U \in m(E^*, E^*)$, and hence,

$U(t * g) = t * Ug = Ut * g,$

Now Ut, being a trigonometric polynomial, lies in E. So,

(3.4) $Ut(g^{\vee}) = Ut * g(0) = T(t * g), \quad \forall g \in E^*.$

Now, (3.3) yields

$| U(g^{\vee}) | \leq \| T \| \| t \|_E, \| g^{\vee} \|_{E^*} \qquad \forall g \in E^*,$

which shows that for every trigonometric polynomial t,

$\| Ut \|_E \leq \| T \| \| t \|_E.$

Since C^∞ is dense in E, U can be uniquely extended to E and

$\| Uf \|_E \leq \| T \| \| f \|_E, \qquad \forall f \in E.$

Hence, $U \in m(E,E)$ and (3.4) will be true even if t is replaced by any $f \in E$.

$Uf * g(0) = T(f * g)$ for all $f \in E$ and $g \in E^*$,

Hence, Its follows that-

$T(h) = \sum Uf_i * g_i(0),$ whenever $\sum f_i * g_i$ is a representation of an element $h \in A(E,E)$. Since the mapping $U \to T$ is norm preserving, it is one-to-one and proves the main part of the theorem.

Now suppose that

$\phi_U(h) = \sum Uf_i * g_i(0)$

Where $\sum f_i * g_i$ is a representation of an element h of $A(E,E)$. For each $x \in G$, $T_x \in m(E,E)$, and therefore, $\phi_{T_x}(h) = h(-x)$ now, if $\phi_{T_x}(h) = 0$ for every $x \in G$, then $h = 0$ and hence $T(h) = 0$. For all T$\in (A(E,E))^*$ An application of the Hahn–Banach theorem shows that the span of $\{\phi_{T_x} : x \in G\}$ is dense in $A(E,E)^*$ with the weak topology. In view of the first assertion, this proves that the span of translation operators is weakly dense in $m(E,E)$.

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