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GLOBAL STABILITY ANALYSIS OF TWO MUTUALLY INTERACTING SPECIES PAIR WITH MONOD TYPE-VARIABLE COEFFICIENT OF ONE OF THE SPECIES

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Abstract: *In the present investigation, the global stability analysis of a two species ecological mutualism monod model is presented by constructing a suitable Liapunov's function for the co-existent equilibrium state.*

Keywords: *Equilibrium States, Mutualism, Coexistent state, Global Stability, Liapunov's function.*

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1 INTRODUCTION

Lotka [6] and Volterra [7] introduced the prey- predator models. K. Laxminarayan and N.Ch.Pattabhiramacharyulu [1, 2, 3] examined the stability of two species Prey-Predator models and derived some threshold theorems on the quasi-linear basic balancing equations. Local stability analysis for a two-species ecological mutualism model has been presented by the present author et al [4, 5].

The present investigation is devoted to establish the global stability of the co-existent equilibrium state of the above said model by employing a properly constructed Liapunov's function.

2 LIAPUNOV'S STABILITY ANALYSIS

A.M. Liapunov introduced an efficient method in 1892 to study the global stability of equilibrium points in case of linear and non-linear systems. The method called Liapunov's method is based on the constructing a scalar function called Liapunov's function. This method yields stability information directly without solving the differential equations involved in the system. Hence it is also called Liapunov's direct method to detect the criteria for global stability. This tool is being employed efficiently in diverse areas such as theory of control systems, dynamical systems, systems with time tag, power system analysis, time varying non-linear feedback systems, multi species ecological systems and so on.

The stability behaviour of solutions of linear and weakly non-linear systems is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighbourhood of operating point i.e. local stability. Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. Hence, the stability behaviour of a physical system is curbed by these limitations.

If the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable. This idea was generalized by Liapunov to study stability problems in a broader context.

Consider an autonomous system

$$\frac{dx}{dt} = F(x, y) \tag{1}$$

$$\frac{dy}{dt} = G(x, y)$$

Assume that this system has an isolated critical point (0, 0). Consider a function E(x, y) possessing continuous partial derivatives along the path of (1).

This path is represented by C = [x (t), y (t)] in the parametric form. E(x, y) can be regarded as a function of t along C with rate of change

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} = \frac{\partial E}{\partial x}\,\mathrm{F} + \frac{\partial E}{\partial y}\,\mathrm{G} \qquad (2)$$

Definitions:

1. E(x, y) is said to be positive definite if E(x, y) > 0 for all (x, y) not equal to (0, 0)
2. E(x, y) is said to be positive semi-definite if E(x, y) > 0 and E(x, y)=0
3. E(x, y) is said to be negative definite if E(x, y) < 0
4. E(x, y) is said to be negative semi-definite if E(x, y) < 0 and E(x, y)=0

A positive definite function E(x, y) with the property that (2) is negative semi–definite is called a Liapunov's function for the system (1). The following theorem is the Liapunov's basic discovery.

Theorem: If there exists a Liapunov's function E(x, y) for the system (1), then the critical point (0, 0) is stable. Furthermore, if this function has additional property that the function (2) is negative definite, then the critical point (0, 0) is asymptotically stable.

A Mathematical Monod model of a two species ecological mutualism is given by the differential equation pair employing the following notation:

N_1, N_2: The populations of the first species (S_1) and the second species (S_2), respectively at time t.

$K_i = \frac{a_i}{a_{ii}}$: Carrying capacities of S_i, i = 1, 2 (these parameters characterize the amount of resources available for the consumption exclusively for the two species.)

Further both the variables N_1 and N_2 are non-negative and the model parameters a_1, a_2, a_{11}, a_{22}, a_{21} are assumed to be non-negative constants.

Equations for the growth rate of the first species (S_1) and the second species (S_2) can be written as follows:

(i) Equation for the growth rate of the first Species (S_1):

$$\frac{dN_1}{dt} = N_1\left[a_{11}\left(K_1 - N_1\right) + F\left(N_2\right)\right] \qquad (3)$$

In the equation (3), the function *F (N_2)* is the characteristic of the N_1 with respect to N_2 with the conditions:

(i) $F(N_2)$ is bounded throughout and

(ii) $F(0) = 0$

The characteristic model considered in this chapter is a two parameter model of the monod type as given by Kapur [8].

$$F(N_2) = \frac{\alpha N_2}{\beta + N_2}$$

Here $\alpha = F(\infty) > 0$ is a parameter characteristic of species (S_1). Further $\beta(\neq 0)$ is another parameter signifying the strength of the one of the other species (S_2). The mutualism is strong or weak according as $\beta > 0$ or $\beta < 0$ and the interaction would be neutral when $\beta = 0$.

(ii) Equation for the growth rate of the second species (S_2):

$$\frac{dN_2}{dt} = N_2\left[K_2 a_{22} - a_{22}N_2 + a_{21}N_1\right] \tag{4}$$

The equilibrium states for this system are

Both washed out state : $E_1 = (0,0)$ (5)

N_2 Only washed out state: $E_2 = (0, K_2)$ (6)

N_1 Washed out state : $E_3 = (K_1, 0)$ (7)

Coexistent state :

$$E_4 = \left(\frac{K_1 a_{11} a_{22} + \dfrac{\alpha K_2 a_{22}}{\beta + K_2}}{a_{11}a_{22} - \dfrac{\alpha}{\beta + K_2} a_{21}}, \frac{K_1 a_{11} a_{21} + K_2 a_{11} a_{22}}{a_{11}a_{22} - \dfrac{\alpha}{\beta + K_2} a_{21}} \right) \tag{8}$$

The local stability analysis of these equilibrium states were investigated in [8]. It is observed that

(a) E_1 is clearly unstable.

(b) E_2 is unstable.

(c) E_3 is unstable.

(d) E_4 is stable.

4 LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY OF THE COEXISTENT STATE E_4:

We shall construct a Liapunov's function and describe the global stability of a system concerning two mutually interacting species with monod type-variable coefficient of one of the species.

Basically we consider the equations:

$$\frac{dN_1}{dt} = N_1\left[a_1 - a_{11}N_1 + \frac{\alpha N_2}{\beta + N_2}\right] \tag{9}$$

$$\frac{dN_2}{dt} = N_2\left[a_2 - a_{22}N_2 + a_{21}N_1\right] \tag{10}$$

The linearized basic equations are

$$\frac{du_1}{dt} = -a_{11}\overline{N}_1 u_1 + \frac{\alpha}{\beta + \overline{\overline{N}}_2}\left(1 - \frac{\overline{N}_2}{\beta + \overline{\overline{N}}_2}\right)\overline{N}_1 u_2 \tag{11}$$

$$\frac{du_2}{dt} = a_{21}\overline{N}_2 u_1 - a_{22}\overline{N}_2 u_2 \tag{12}$$

The characteristic equation is

$$\lambda^2 + \left(a_{11}\overline{N}_1 + a_{22}\overline{N}_2\right)\lambda + \left\{a_{11}a_{22} - a_{21}\frac{\alpha}{\beta + \overline{\overline{N}}_2}\left(1 - \frac{\overline{N}_2}{\beta + \overline{\overline{N}}_2}\right)\overline{N}_1\overline{N}_2\right\} = 0 \tag{13}$$

Equation (13) is of the form $\lambda^2 + p\lambda + q = 0$

where $p = a_{11}\overline{N}_1 + a_{22}\overline{N}_2 > 0$ (14)

$$q = \left[a_{11}a_{22} - a_{21}\frac{\alpha}{\beta + \overline{\overline{N}}_2}\left(1 - \frac{\overline{N}_2}{\beta + \overline{\overline{N}}_2}\right)\right]\overline{N}_1\overline{N}_2$$

$$= \left[a_{11}a_{22} - a_{21}\frac{\alpha}{\beta + \overline{\overline{N}}_2} + \frac{a_{21}\alpha\overline{N}_2}{\beta + \overline{\overline{N}}_2}\right]\overline{N}_1\overline{N}_2 > 0 \qquad \left(\because a_{11}a_{22} > \frac{\alpha a_{21}}{\beta + \overline{\overline{N}}_2}\right) \tag{15}$$

Therefore the conditions for Liapunov's function are satisfied

Now we define

$E(u_1, u_2) = \frac{1}{2}(a u_1^2 + 2b\, u_1 u_2 + c u_2^2)$ (16)

where

$$a = \frac{\left(a_{21}\overline{N}_2\right)^2 + \left(a_{22}\overline{N}_2\right)^2 + q}{D} \tag{17}$$

$$b = \frac{\left[\frac{\alpha}{\beta+\overline{N_2}}\left(1-\frac{\overline{N_2}}{\beta+\overline{N_2}}\right)a_{22}+a_{11}a_{21}\right]\overline{N_1}\,\overline{N_2}}{D} \tag{18}$$

$$c = \frac{\left(a_{11}\overline{N_1}\right)^2+\left[\frac{\alpha}{\beta+\overline{N_2}}\left(1-\frac{\overline{N_2}}{\beta+\overline{N_2}}\right)\overline{N_1}\right]^2+q}{D} \tag{19}$$

$$D = pq = (a_{11}\overline{N_1} + a_{22}\overline{N_2})\left(a_{11}a_{22}-a_{21}\frac{\alpha}{\beta+\overline{N_2}}+\frac{a_{21}\alpha\overline{N_2}}{\beta+\overline{N_2}}\right)\overline{N_1}\,\overline{N_2} \tag{20}$$

From equations (14) and (15) it is clear that D > 0 and a > 0

Also

$$D^2(ac-b^2) = D^2\left\{\left(\frac{(a_{21}\overline{N_2})^2+(a_{22}\overline{N_2})^2+q}{D}\right)\left(\frac{(a_{11}\overline{N_1})^2+\left(\frac{\alpha}{\beta+\overline{N_2}}\left(1-\frac{\overline{N_2}}{\beta+\overline{N_2}}\right)\overline{N_1}\right)^2+q}{D}\right) - \left[\frac{\frac{\alpha}{\beta+\overline{N_2}}\left(1-\frac{\overline{N_2}}{\beta+\overline{N_2}}\right)a_{22}\overline{N_1}\,\overline{N_2}}{D}\right]^2\right\} \tag{21}$$

$$= \left(a_{21}^2\overline{N_2}^2+a_{22}^2\overline{N_2}^2+q\right)\left(a_{11}^2\overline{N_1}^2+\frac{\alpha^2}{\left(\beta+\overline{N_2}\right)^2}\frac{\beta^2}{\left(\beta+\overline{N_2}\right)^2}\overline{N_1}^2+q\right) - \frac{\alpha^2}{\left(\beta+\overline{N_2}\right)^2}\frac{\beta^2}{\left(\beta+\overline{N_2}\right)^2}a_{22}^2\overline{N_1}^2\overline{N_2}^2$$

$$a_{11}^2a_{21}^2\overline{N_1}^2\overline{N_2}^2+2a_{11}a_{21}a_{22}\frac{\alpha\beta}{\left(\beta+\overline{N_2}\right)^2}\overline{N_1}^2\overline{N_2}^2 \tag{22}$$

$$= a_{11}^2a_{22}^2\overline{N_1}^2\overline{N_2}^2+\frac{\alpha^2}{\left(\beta+\overline{N_2}\right)^2}\frac{\beta^2}{\left(\beta+\overline{N_2}\right)^2}(a_{21}^2\overline{N_1}^2\overline{N_2}^2+\overline{N_1}^2q)+(a_{21}^2+a_{22}^2)q\overline{N_2}^2+qa_{11}^2\overline{N_1}^2+q^2>0 \tag{23}$$

$\Rightarrow D^2(ac - b^2) > 0$

Therefore the function E (u_1, u_2) at (16) is positive definite.

Further
$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt} = \left(au_1+bu_2\right)\left(-a_{11}\overline{N_1}u_1+\frac{\alpha}{\beta+\overline{N_2}}\left(1-\frac{\overline{N_2}}{\beta+\overline{N_2}}\right)\overline{N_1}u_2\right) + \left(bu_1+cu_2\right)\left(a_{21}\overline{N_2}u_1-a_{22}\overline{N_2}u_2\right) \tag{25}$$

By substituting values of a, b and c from equations (17), (18) and (19) in (25) we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -(u_1^2+u_2^2) \tag{26}$$

which is clearly negative definite.

So E (u_1, u_2) is a Liapunov function for the linear system.

Next we prove that E (u_1, u_2) is also a Liapunov function for the non-Linear system also.

If f_1 and $f_2$are two functions in N_1 and N_2 defined by

$$f_1(N_1, N_2) = N_1\left[a_1 - a_{11}N_1 + \frac{\alpha N_2}{\beta + N_2}\right] \quad (27)$$

$$f_2(N_1, N_2) = N_2\left[a_2 - a_{22}N_2 + a_{21}N_1\right] \quad (28)$$

we now have to show that $\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite.

Putting $N_1 = \overline{N_1} + u_1$ and $N_2 = \overline{N_2} + u_2$ in (9) and (10) we get

$$\frac{du_1}{dt} = \left(\overline{N_1} + u_1\right)\left(\left(a_1 - a_{11}(\overline{N_1} + u_1\right) + \frac{\alpha(\overline{N}_2 + u_2)}{\beta + (\overline{N}_2 + u_2)}\right)$$

$$= \left(\overline{N_1} + u_1\right)a_1 - a_{11}(\overline{N}_1 + u_1)^2 + \frac{\alpha}{\beta + \overline{N_2}}\left[1 + \frac{u_2}{\beta + N_2}\right]^{-1}(\overline{N}_2 + u_2)(\overline{N}_1 + u_1)$$

$$= a_1\overline{N_1} + a_1 u_1 - a_{11}(\overline{N_1}^2 + 2\overline{N_1}u_1 + u_1^2) + \frac{\alpha}{\beta + \overline{N_2}}\left(1 - \frac{u_2}{\beta + \overline{N_2}}\right)\left(\overline{N}_1\overline{N_2} + \overline{N}_1 u_2 + \overline{N}_2 u_1 + u_1 u_2\right)$$

$$= -a_{11}\overline{N_1}u_1 + \left(\frac{\alpha\overline{N_1}}{\beta + \overline{N_2}} - \frac{\alpha\overline{N_1}\,\overline{N_2}}{\left(\beta + \overline{N_2}\right)^2}\right)u_2 - a_{11}u_1^2 - \frac{\alpha\overline{N_1}u_2^2}{\left(\beta + \overline{N_2}\right)^2} + \left(\frac{\alpha}{\beta + \overline{N_2}} - \frac{\alpha\overline{N_2}}{\left(\beta + \overline{N_2}\right)^2}\right)u_1u_2 - \frac{\alpha}{\left(\beta + \overline{N_2}\right)^2}u_1u_2^2$$

$$\Rightarrow f_1(u_1, u_2) = \frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 + \left(\frac{\alpha\overline{N_1}}{\beta + \overline{N_2}} - \frac{\alpha\overline{N_1}\,\overline{N_2}}{\left(\beta + \overline{N_2}\right)^2}\right)u_2 + F(u_1, u_2) \quad (29)$$

where

$$F(u_1, u_2) = -a_{11}u_1^2 - \frac{\alpha\overline{N_1}}{\beta + \overline{N_2}}u_2^2 + \left(\frac{\alpha}{\beta + \overline{N_2}} - \frac{\alpha\overline{N_2}}{\left(\beta + \overline{N_2}\right)^2}\right)u_1u_2 - \frac{\alpha}{\left(\beta + \overline{N_2}\right)^2}u_1u_2^2$$

Also

$$\frac{du_2}{dt} = \left(\overline{N_2} + u_2\right)\left(a_2 - a_{22}\left(\overline{N_2} + u_2\right) + a_{21}\left(\overline{N_1} + u_1\right)\right)$$

$$\Rightarrow f_2(u_1, u_2) = \frac{du_2}{dt} = a_{21}\overline{N_2}u_1 - a_{22}\overline{N_2}u_2 + G(u_1, u_2) \quad (30)$$

where $G(u_1, u_2) = -a_{22}u_2^2$

From (16)

$$\frac{\partial E}{du_1} = au_1 + bu_2 \quad (31)$$

$$\frac{\partial E}{du_2} = bu_1 + cu_2 \tag{32}$$

Now

$$\frac{\partial E}{du_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = (au_1 + bu_2)\left[-a_{11}\overline{N_1}u_1 + \left(\frac{\alpha \overline{N_1}}{\beta + \overline{N_2}} - \frac{\alpha \overline{N_1}\,\overline{N_2}}{(\beta + \overline{N_2})^2}\right)u_2 + F(u_1, u_2)\right]$$

$$+ (b\,u_1 + c\,u_2)[\,a_{21}\,\overline{N_2}\,u_1 - a_{22}\,\overline{N_2}\,u_2 + G\,(u_1, u_2)] \tag{33}$$

$$\frac{\partial E}{du_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = \left\{(au_1 + bu_2)\left[-a_{11}\overline{N_1}u_1 + \frac{\alpha}{\beta + \overline{N_2}}\left(1 - \frac{\overline{N_2}}{\beta + \overline{N_2}}\right)\overline{N_1}u_2\right]\right.$$

$$\left. + (bu_1 + cu_2)(a_{21}\overline{N_2}u_1 - a_{22}\overline{N_2}u_2)\right\} + [(au_1 + bu_2)\,F\,(u_1, u_2) + (bu_1 + cu_2)\,G\,(u_1, u_2)]$$

$$\tag{34}$$

From (26)

$$\frac{\partial E}{du_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -\,(u_1^2 + u_2^2) + (au_1 + bu_2)\,F(u_1, u_2) + (bu_1 + cu_2)\,G\,(u_1, u_2) \tag{35}$$

By introducing polar co-ordinates $u_1 = r\cos\theta$, $u_2 = r\sin\theta$ we can write the equation (35) as

$$\frac{\partial E}{du_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -\,(r^2) + r\,\{[a\cos\theta + b\sin\theta\,]\,F(u_1, u_2) + [b\cos\theta + c\sin\theta\,]\,G(u_1, u_2) \tag{36}$$

Let us denote largest of the numbers |a|, |b|, |c| by K.

Our assumptions imply that | F (u_1, u_2) | < $\frac{r}{6K}$ and | G (u_1, u_2)| < $\frac{r}{6K}$ for all sufficiently small

r > 0.

So $\frac{\partial E}{du_1} f_1 + \frac{\partial E}{\partial u_2} f_2 < -r^2 + \frac{4Kr^2}{6K} = -\frac{r^2}{3} < 0$ (37)

Thus the function E (u_1, u_2) is positive definite with the condition that

$\frac{\partial E}{du_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite

∴ The equilibrium state **E_4** is **"asymptotically stable".**

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