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## SOME THRESHOLD THEOREMS FOR A PREY-PREDATOR MODEL WITH AN OPTIMAL HARVESTING OF THE PREDATOR

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**Abstract:** *The present paper is devoted to derive some threshold theorems for a two species model comprising a prey and a predator. Predator is provided with a limited resource of food in addition to the prey and the predator is harvested under optimal conditions. In consonance with the principle of competitive exclusion Gauss, three theorems and ten lemmas has been derived. The model is characterized by a couple of first order non-linear ordinary differential equations.*

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## 1 INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [3], Kushing [4], Paul colinvaux [5], Kapur [6] etc. The ecological interactions can be broadly classified as Prey – predation, Competition, Commensalism, Ammensalism, Neutralism and so on. N.C.Srinivas [7] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later, Lakshminarayan and Pattabhi Ramacharyulu [8] studied some threshold theorems for a Prey-predator model harvesting. Recently, the present author et al [9-12] investigated mutualism between two species.

## 2 BASIC EQUATIONS

The model equations for a two species prey-predator system are given by the following system of non-linear ordinary differential equations employing the following notation:

$N_1$  and  $N_2$  are population of the prey and predator,

$a_1$  and  $a_2$  are the rates of natural growth of the prey and predator,

$\alpha_{11}$  is rate of decrease of the prey due to insufficient food,

$\alpha_{12}$  is rate of decrease of the prey due to successful attacks by the predator,

$\alpha_{22}$  is rate of decrease of the predator due to insufficient food other than the prey,

$\alpha_{21}$  is rate of increase of the predator due to successful attacks on the prey,

$q_2$  is the catchability co-efficient of predator,

$E$  is the harvesting effort and  $q_2EN_2$  is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis.

Further both the variables  $N_1$  and  $N_2$  are non-negative and the model parameters  $a_1, a_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, q_2, E$  and  $(a_2 - q_2E)$  are assumed to be non-negative constants. Employing the above terminology, the model equations for a two species prey-predator system is given by the following system of non-linear ordinary differential equations.



(i) Equation for the growth rate of prey species ( $N_1$ ):

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 \quad (2.1)$$

(ii) Equation for the growth rate of predator species ( $N_2$ ):

$$\begin{aligned} \frac{dN_2}{dt} &= a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_1N_2 - q_2EN_2 \\ \Rightarrow \frac{dN_2}{dt} &= (a_2 - q_2E)N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_1N_2 \end{aligned} \quad (2.2)$$

### 3 EQUILIBRIUM STATES

The system under investigation has four equilibrium states:

I. The fully washed out state with the equilibrium point  $\bar{N}_1 = 0; \bar{N}_2 = 0$  (3.1)

II. The state in which, only the predator survives and the prey is washed out.

$$\text{The equilibrium point is } \bar{N}_1 = 0; \bar{N}_2 = \frac{(a_2 - q_2E)}{\alpha_{22}} \quad (3.2)$$

III. The state in which, only the prey survives and the predator is washed out

$$\text{The equilibrium point is } \bar{N}_1 = \frac{a_1}{\alpha_{11}}; \bar{N}_2 = 0 \quad (3.3)$$

IV. The co-existent state (**normal steady state**). The equilibrium point is

$$\bar{N}_1 = \frac{a_1\alpha_{22} - (a_2 - q_2E)\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}; \quad \bar{N}_2 = \frac{(a_2 - q_2E)\alpha_{11} + a_1\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \quad (3.4)$$

This state would exist only when  $a_1\alpha_{22} > (a_2 - q_2E)\alpha_{12}$

### 4 THRESHOLD THEOREMS

In consonance with the principle of competitive exclusion, Gauss [13] three Threshold Theorems on the basic equations (2.1) & (2.2), one for each of the above three not-fully washed equilibrium states has been deduced. The equations are:

$$\frac{dN_1}{dt} = \frac{a_1N_1}{k_1} \{k_1 - N_1 - \beta_1N_2\}, \quad \frac{dN_2}{dt} = \frac{(a_2 - q_2E)N_2}{k_2} \{k_2 - N_2 - \beta_2N_1\} \quad (4.1)$$

where

$$k_1 = \frac{a_1}{\alpha_{11}}; k_2 = \frac{(a_2 - q_2E)}{\alpha_{22}}; \beta_1 = \frac{\alpha_{12}}{a_1} \quad \text{and} \quad \beta_2 = -\frac{\alpha_{21}}{(a_2 - q_2E)}$$

**Theorem 1:** Principle of Competitive Exclusion for Equilibrium State II:



When  $k_1 > k_2$ , then every solution  $N_1(t), N_2(t)$  of (4.1) approaches the equilibrium solution  $N_1 = k_1, N_2 = 0$  as  $t$  approaches infinity. In other words, if species 1 and 2 are nearly identical and the microcosm can support more members of species 1 than species 2, then species 2 will ultimately become extinct.

**Proof:** The first step in proving this is to show that  $N_1(t)$  and  $N_2(t)$  can never become negative. To this end, observe that

$$N_1(t) = \frac{k_1 N_1(0)}{N_1(0) + (k_1 - N_1(0))e^{-a_1 t}} \text{ and } N_2(t) = 0 \quad (4.2)$$

is a solution of (4.1) for any choice of  $N_1(0)$ . The orbit of this solution in the  $N_1 - N_2$  plane is the point  $(0, 0)$  for  $N_1(0) = 0$ ; the line  $0 < N_1 < k_1, N_2 = 0$  for  $0 < N_1(0) < k_1$ ; the point  $(k_1, 0)$  for  $N_1(0) = k_1$ ; and the line  $k_1 < N_1 < \infty, N_2 = 0$  for  $N_1(0) > k_1$ . Thus the  $N_1$  axis, for  $N_1 \geq 0$ , is the union of four distinct orbits of (4.1). Similarly, the  $N_2$  axis, for  $N_2 \geq 0$ , is the union of four distinct orbits of (4.1). This implies that  $N_1(t), N_2(t)$  of (4.1) which starts in the first quadrant ( $N_1(t) > 0, N_2 > 0$ ) of the  $N_1 - N_2$  plane must remain there for all future time.

The second step is to split the first quadrant into regions in which both  $\frac{dN_1}{dt}$  and  $\frac{dN_2}{dt}$  have fixed signs. This is accomplished in the following manner.

Let  $l_1$  and  $l_2$  be the lines

$$(k_1 - N_1 - \beta_1 N_2) = 0 \quad (4.3)$$

and

$$(k_2 - N_2 - \beta_2 N_1) = 0 \quad (4.4)$$

These lines are non-parallel and non-intersecting in  $N_1 - N_2$  plane respectively (Ref.Fig.1).

Observe that  $\frac{dN_1}{dt}$  is negative if  $(N_1, N_2)$  lies above  $l_1$  and positive if  $(N_1, N_2)$  lies below  $l_1$ .

Similarly,  $\frac{dN_2}{dt}$  is negative if  $(N_1, N_2)$  lies above  $l_2$  and positive if  $(N_1, N_2)$  lies below  $l_2$ .

Thus the two lines  $l_1$  and  $l_2$  split the first quadrant of the  $N_1 - N_2$  plane into three regions



in which both  $\frac{dN_1}{dt}$  and  $\frac{dN_2}{dt}$  have fixed signs. Both  $N_1(t)$ ,  $N_2(t)$  increases with time (along any solution of (4.1) in region I;  $N_1(t)$  increases and  $N_2(t)$  decreases with time in region II; and both  $N_1(t)$  and  $N_2(t)$  decrease with time in region III. This is illustrated in Fig.1.

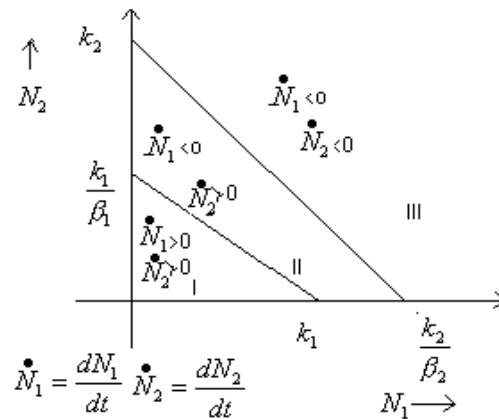


Fig.1

Finally we require the following three lemmas for establishing the threshold theorems.

**Lemma 1:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region I at time  $t = t_0$  must leave this region I at some latter instant of time (Fig.1).

**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) remain in region I for all time  $t \geq t_0$ . This implies that both  $N_1(t)$  and  $N_2(t)$  are monotonic increasing functions of time for  $t \geq t_0$ , with  $N_1(t)$  and  $N_2(t)$  less than  $k_2$ . Consequently both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This, in turn, implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now the only equilibrium points of (4.1) are  $(0, 0)$ ,  $(k_1, 0)$ ,  $(0, k_2)$  and  $(\xi, \eta)$  obviously cannot equal any of these three points. We conclude, therefore, that any solution  $N_1(t), N_2(t)$  of (4.1) which starts in region I must leave this region at a later time.

**Lemma 2:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region II at time  $t = t_0$  will remain in this region for all future time  $t \geq t_0$ , and ultimately approach the equilibrium solution  $N_1 = k_1, N_2 = 0$  (Fig.1).



**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) leaves region II at time  $t = t^*$ . Then either  $\frac{dN_1}{dt}(t^*)$  or  $\frac{dN_2}{dt}(t^*)$  is zero, since the only way a solution of (4.1) can leave region II

is by crossing  $l_1$  or  $l_2$ . Assume that  $\frac{dN_1}{dt}(t^*) = 0$ . Differentiation both sides of the first equation of (4.4.1) with respect to  $t$  and setting  $t = t^*$  gives

$$\frac{d^2 N_1(t^*)}{dt} = \frac{-a_1 \beta_1 N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt} \quad (4.5)$$

This quantity is positive. Hence  $N_1(t)$  has a minimum at  $t = t^*$ . However, this is impossible, since  $N_1(t)$  is increasing whenever a solution of  $N_1(t), N_2(t)$  of (4.4.1) is in region II.

Similarly, if  $\frac{dN_2}{dt}(t^*) = 0$ , then

$$\frac{d^2 N_2(t^*)}{dt} = \frac{-(a_2 - q_2 E) \beta_2 N_2(t^*)}{k_2} \frac{dN_1(t^*)}{dt} \quad (4.6)$$

This quantity is negative, implying that  $N_2(t)$  has a maximum at  $t = t^*$ , but this is impossible, since  $N_2(t)$  is decreasing whenever a solution  $N_1(t), N_2(t)$  of (4.1) is in region II.

The previous argument shows that any solution  $N_1(t), N_2(t)$  of (4.1) which starts in region II at time  $t = t_0$  will remain in region II for all future time  $t \geq t_0$ . This implies that  $N_1(t)$  is monotonic increasing and  $N_2(t)$  is monotonic decreasing for  $t \geq t_0$ ; with  $N_1(t) < k_1$  and  $N_2(t) > k_2$ . Consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This in turn, implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$  or  $(0, k_2)$ . Consequently,  $(\xi, \eta) = (k_1, 0)$  and this proves Lemma 2.

**Lemma 3:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region III at time  $t \geq t_0$  and remains there for all future time must approach the equilibrium solution  $N_1(t) = k_1, N_2(t) = 0$  as  $t$  approaches infinity (Fig.1).

**Proof:** If a solution  $N_1(t), N_2(t)$  of (4.1) remains in region III for  $t \geq t_0$ , then both  $N_1(t)$  and  $N_2(t)$  are monotonic decreasing functions of time for  $t \geq t_0$ , with  $N_1(t) > k_1$  and  $N_2(t) > k_2$ , consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity.



This, in turn implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now,  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$  or  $(0, k_2)$ . Consequently  $(\xi, \eta) = (k_1, 0)$ .

**Proof of Theorem:** Lemmas 1 and 2 state that every solution  $(N_1(t), N_2(t))$  of (4.1) which starts in region I or II at time  $t = t_0$  must approach the equilibrium solution  $N_1 = k_1, N_2 = 0$  as  $t$  approaches infinity. Similarly, Lemma 3 shows that every solution  $(N_1(t), N_2(t))$  of (4.1) which starts in region III at time  $t = t_0$  and remains there for all future time must also approach equilibrium solution  $N_1 = k_1, N_2 = 0$ . Next, observe that any solution  $(N_1(t), N_2(t))$  of (4.1) which starts on  $l_1$  or  $l_2$  would soon enter region II. Finally, if a solution  $(N_1(t), N_2(t))$  of (4.1) leaves region III, then it must cross the line  $l_1$  and immediately afterwards enters region II. Lemma 2 then forces the solution to approach the equilibrium solution  $N_1 = k_1, N_2 = 0$ . This is illustrated in the Fig.2

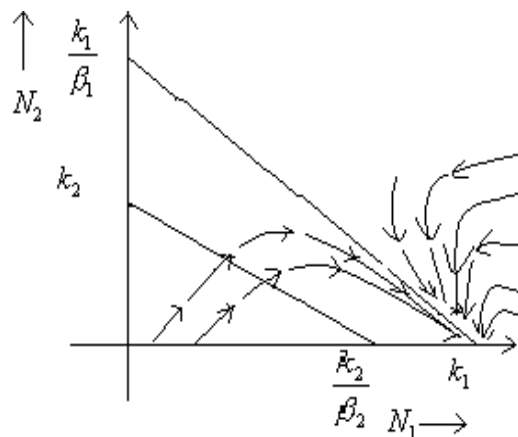


Fig.2

**Theorem 2:** Principle of Competitive Exclusion for Equilibrium State III:

$$\bar{N}_1 = \frac{a_1}{\alpha_{11}}; \bar{N}_2 = 0$$

When  $k_1 < k_2$ , then every solution  $N_1(t), N_2(t)$  of (4.1) approaches the equilibrium solution  $N_1 = 0, N_2 = k_2$  as  $t$  approaches infinity. In other words, if species 1 and 2 are nearly identical and the microcosm can support more members of species 1 than species 2, then species 2 will ultimately become extinct.

**Proof:** The first step in our proof is to show that  $N_1(t)$  and  $N_2(t)$  can never become negative. To this end, we observe that



$$N_1 = 0 \text{ and } N_2(t) = \frac{k_2 N_2(0)}{N_2(0) + (k_2 - N_2(0))e^{-(a_2 - q_2 E)t}} \quad (4.7)$$

is a solution of (4.1) for any choice of  $N_2(0)$ . The orbit of this solution in the  $N_1 - N_2$  plane is the point  $(0, 0)$  for  $N_2(0)=0$ ; the line  $0 < N_1 < k_1$ ,  $N_1(=0)$  for  $0 < N_2(0) < k_2$ ; the point  $(0, k_2)$  for  $N_2(0) = k_2$ ; and the line  $k_2 < N_2 < \infty$ ,  $N_1 = 0$  for  $N_2(0) > k_2$ . Thus the  $N_2$  axis, for  $N_2 \geq 0$ , is the union of four distinct orbits of (4.1). Similarly, the  $N_1$  axis, for  $N_1 \geq 0$ , is the union of four distinct orbits of (4.1). This implies that  $N_1(t), N_2(t)$  of (4.1) which starts in the first quadrant ( $N_1(t) > 0, N_2 > 0$ ) of the  $N_1 - N_2$  plane must remain there for all future time.

The second step in our proof is to split the first quadrant into regions in which both  $\frac{dN_1}{dt}$  and  $\frac{dN_2}{dt}$  have fixed signs. This is accomplished in the following manner.

Let  $l_1$  and  $l_2$  be the lines  $(k_1 - N_1 - \beta_1 N_2) = 0$  and  $(k_2 - N_2 - \beta_2 N_1) = 0$  respectively. Observe that  $\frac{dN_1}{dt}$  is negative if  $(N_1, N_2)$  lies above  $l_1$  and positive if  $(N_1, N_2)$  lies below  $l_1$ . Similarly,  $\frac{dN_2}{dt}$  is negative if  $(N_1, N_2)$  lies above  $l_2$  and positive if  $(N_1, N_2)$  lies below  $l_2$ . Thus the two parallel lines  $l_1$  and  $l_2$  split the first quadrant of the  $N_1 - N_2$  plane into three regions in which both  $\frac{dN_1}{dt}$  and  $\frac{dN_2}{dt}$  have fixed signs. Both  $N_1(t), N_2(t)$  increases with time along any solution of (4.1) in region *I*;  $N_1(t)$  increases and  $N_2(t)$  decreases with time in region *II*; and both  $N_1(t)$  and  $N_2(t)$  decrease with time in region *III* (Ref. Fig. 3). We require the following three lemmas.



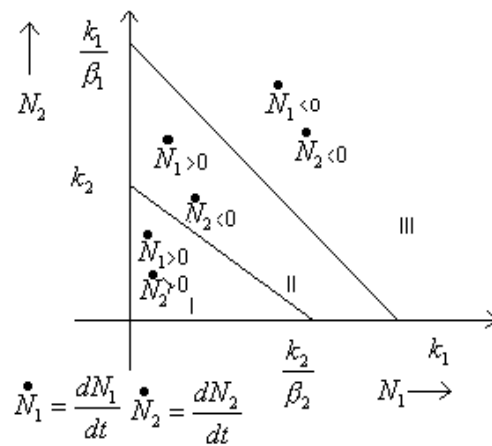


Fig.3

**Lemma 4:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region I at time  $t = t_0$  must leave this region I at some latter time. (Fig.3)

**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) remain in region I for all time  $t \geq t_0$ . This implies that both  $N_1(t)$  and  $N_2(t)$  are monotonic increasing functions of time for  $t \geq t_0$ , with  $N_1(t)$  and  $N_2(t)$  less than  $k_2$ . Consequently both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This, in turn, implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now the only equilibrium points of (4.1) are  $(0, 0), (k_1, 0), (0, k_2)$  and obviously  $(\xi, \eta)$  cannot equal any of these three points. We conclude, therefore, that any solution  $N_1(t), N_2(t)$  of (4.1) which starts in region I must leave this region at a later time.

**Lemma 5:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region II at time  $t = t_0$  will remain in this region for all future time  $t \geq t_0$ , and ultimately approach the equilibrium solution  $N_1 = 0, N_2 = k_2$  (Fig.3).

**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) leaves region II at time  $t = t^*$ . Then either  $\frac{dN_1}{dt}(t^*)$  or  $\frac{dN_2}{dt}(t^*)$  is zero, since the only way a solution of (4.1) can leave region II

is by crossing  $l_1$  or  $l_2$ . Assume that  $\frac{dN_1}{dt}(t^*) = 0$ . Differentiation both sides of the first equation of (4.4.1) with respect to  $t$  and setting  $t = t^*$  gives

$$\frac{d^2 N_1(t^*)}{dt} = \frac{-a_1 \beta_1 N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt} \quad (4.8)$$



This quantity is positive. Hence  $N_1(t)$  has a minimum at  $t = t^*$ . However, this is impossible, since  $N_1(t)$  is increasing whenever a solution of  $N_1(t), N_2(t)$  of (4.1) is in region II.

Similarly, if  $\frac{dN_2}{dt}(t^*) = 0$ , then

$$\frac{d^2N_2(t^*)}{dt} = \frac{-(a_2 - q_2E)\beta_2N_2(t^*)}{k_2} \frac{dN_1}{dt}(t^*). \quad (4.9)$$

This quantity is negative, implying that  $N_2(t)$  has a maximum at  $t = t^*$ , but this is impossible, since  $N_2(t)$  is decreasing whenever a solution  $N_1(t), N_2(t)$  of (4.1) is in region II.

The previous argument shows that any solution  $N_1(t), N_2(t)$  of (4.1) which starts in region II at time  $t = t_0$  will remain in region II for all future time  $t \geq t_0$ . This implies that  $N_1(t)$  is monotonic increasing and  $N_2(t)$  is monotonic decreasing for  $t \geq t_0$ ; with  $N_1(t) < k_1$  and  $N_2(t) > k_2$ . Consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This in turn, implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$  or  $(0, k_2)$ . Consequently,  $(\xi, \eta) = (0, k_2)$  and this proves Lemma 5.

**Lemma 6:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region III at time  $t \geq t_0$  and remains there for all future time must approach the equilibrium solution  $N_1(t) = 0, N_2(t) = k_2$  as  $t$  approaches infinity (Fig 3).

**Proof:** If a solution  $N_1(t), N_2(t)$  of (4.1) remains in region III for  $t \geq t_0$ , then both  $N_1(t)$  and  $N_2(t)$  are monotonic decreasing functions of time for  $t \geq t_0$ , with  $N_1(t) > k_1$  and  $N_2(t) > k_2$ , consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This, in turn implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now,  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$  or  $(k_1, 0)$ . Consequently  $(\xi, \eta) = (0, k_2)$ .

**Proof of Theorem:** Lemmas 4 and 5 state that every solution  $N_1(t), N_2(t)$  of (4.1) which starts in region I or II at time  $t = t_0$  must approach the equilibrium solution  $N_1 = 0, N_2 = k_2$  as  $t$  approaches infinity. Similarly, Lemma 6 shows that every solution  $N_1(t), N_2(t)$  of (4.1) which starts in region III at time  $t = t_0$  and remains there for all future time must also approach equilibrium solution  $N_1 = 0, N_2 = k_2$ . Next, observe that any solution  $N_1(t), N_2(t)$



of (4.1) which starts on  $l_1$  or  $l_2$  must immediately afterwards enter region II. Finally, if a solution  $N_1(t), N_2(t)$  of (4.1) leaves region III, then it must cross the line  $l_1$  and immediately afterwards enter region II. Lemma 5 then forces the solution to approach the equilibrium solution  $N_1 = 0, N_2 = k_2$ . This is illustrated in Fig.4.

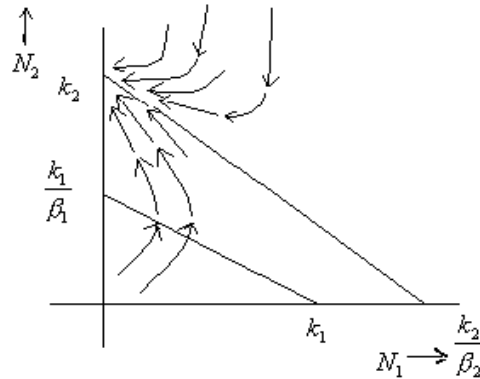


Fig.4

**Theorem 3:** Principle of Competitive Exclusion for Equilibrium State IV:

$$\bar{N}_1 = \frac{a_1\alpha_{22} - (a_2 - q_2E)\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \quad ; \quad \bar{N}_2 = \frac{(a_2 - q_2E)\alpha_{11} + a_1\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}$$

When  $\frac{k_1}{\beta_1} > k_2$  and  $\frac{k_2}{\beta_2} > k_1$ , then every solution of  $N_1(t), N_2(t)$  of (4.1) approaches the equilibrium solution  $N_1(t) = \bar{N}_1 (\neq 0)$  and  $N_2(t) = \bar{N}_2 (\neq 0)$  as  $t$  approaches infinity. In other words, if species 1 and 2 are nearly identical and the microcosm can support both the members of species 1 and 2 depending up on the initial conditions.

**Proof:** The first step in our proof is to show that  $N_1(t)$  and  $N_2(t)$  can never become negative. To this end, observe that

$$N_1(t) = \bar{N}_1 = \frac{a_1\alpha_{22} - (a_2 - q_2E)\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}$$

and

$$N_2(t) = \bar{N}_2 = \frac{(a_2 - q_2E)\alpha_{11} + a_1\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}$$

is a solution of (4.1) for any choice of  $N_1(0)$ . The orbit of this solution in the  $N_1 - N_2$  plane is the point  $(0,0)$  for  $N_1(0) = 0$ ; the line  $0 < N_1 < k_1, N_2 = 0$  for  $0 < N_1(0) < k_1$ ; the point  $(k_1, 0)$  for  $N_1(0) = k_1$ ; and the line  $k_1 < N_1 < \infty, N_2 = 0$  for  $N_1(0) > k_1$ . Thus the  $N_1$  axis,



for  $N_1 \geq 0$  is the union of four distinct orbits of (4.1). Similarly the  $N_2$  axis, for  $N_2 \geq 0$ , is the union of four distinct orbits of (4.1). This implies that all solutions  $N_1(t), N_2(t)$  of (4.1) which start in the first quadrant ( $N_1(t) > 0, N_2 > 0$ ) of the  $N_1 - N_2$  plane must remain there for all future time.

The second step in our proof is to split the first quadrant into regions in which both  $\frac{dN_1}{dt}$  and  $\frac{dN_2}{dt}$  have fixed signs. This is accomplished in the following manner.

Let  $l_1$  and  $l_2$  be the lines  $(k_1 - N_1 - \beta_1 N_2) = 0$  and  $(k_2 - N_2 - \beta_2 N_1) = 0$  respectively and the point of their intersection, is  $(\bar{N}_1, \bar{N}_2)$ . Observe that  $\frac{dN_1}{dt}$  is negative if  $(N_1, N_2)$  lies above the line  $l_1$  and positive if  $(N_1, N_2)$  lies below  $l_1$ . Similarly,  $\frac{dN_2}{dt}$  is negative if  $(N_1, N_2)$  lies above  $l_2$  and positive if  $(N_1, N_2)$  lies below  $l_2$ . Thus the two lines  $l_1$  and  $l_2$  split the first quadrant of the  $N_1 - N_2$  plane into four regions in which both  $\frac{dN_1}{dt}$  and  $\frac{dN_2}{dt}$  have fixed signs.

$N_1(t), N_2(t)$  both increase with time along any solution of (4.1) in region I;

$N_1(t)$  increases and  $N_2(t)$  decreases with time in region II;

$N_1(t)$  decreases and  $N_2(t)$  increases with time in region III and both  $N_1(t)$  and  $N_2(t)$  decrease with time in region IV. In this region both the prey predator compete with each other but do not flourish and at the same time do not get extinct as shown in Fig.5.

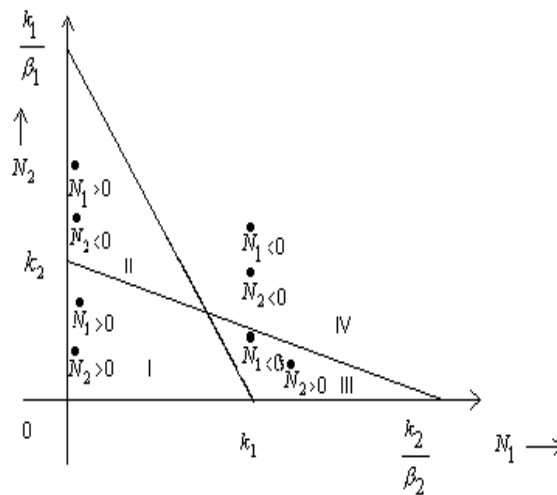


Fig.5



Finally we require the following four lemmas.

**Lemma 7:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region I at time  $t = t_0$  will remain in this region for all future time  $t \geq t_0$ , and ultimately approach the equilibrium solution  $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$  (Fig 5).

**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) leaves region I at time  $t = t^*$ . Then

either  $\frac{dN_1}{dt}(t^*)$  or  $\frac{dN_2}{dt}(t^*)$  is zero, since the only way a solution of (4.1) can leave region I is

by crossing  $l_1$  or  $l_2$ . Assume that  $\frac{dN_1}{dt}(t^*) = 0$ . Differentiation both sides of the first

equation of (4.1) with respect to  $t$  and setting  $t = t^*$  gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-a_1\beta_1N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt} < 0 \quad (4.10)$$

Hence  $N_1(t)$  is monotonic increasing and it has maximum whenever a solution of  $N_1(t), N_2(t)$  of (4.1) is in region I.

Similarly, if  $\frac{dN_2}{dt}(t^*) = 0$ ,

then

$$\frac{d^2N_2(t^*)}{dt} = \frac{-(a_2 - q_2E)\beta_2N_2(t^*)}{k_2} \frac{dN_1(t^*)}{dt} < 0 \quad (4.11)$$

implies that  $N_2(t)$  is monotonic increasing and it has maximum whenever a solution  $N_1(t), N_2(t)$  of (4.1) is in region I.

If a solution  $N_1(t), N_2(t)$  of (4.1) remains in region I for  $t \geq t_0$ , then both  $N_1(t)$  and  $N_2(t)$  are monotonic increasing functions of time for  $t \geq t_0$ , with  $N_1(t) < k_1$  and  $N_2(t) < k_2$ , consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This, in turn implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now,  $(\xi, \eta)$  obviously cannot equal  $(0,0)$ ;  $(k_1, 0)$  or  $(0, k_2)$ . Consequently  $(\xi, \eta) = (\bar{N}_1, \bar{N}_2)$ .

**Lemma 8:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region II at time  $t = t_0$  will remain in this region for all future time  $t \geq t_0$ , and ultimately approach the equilibrium solution  $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$  (Fig 5).



**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) leaves region II at time  $t = t^*$ . Then either  $\frac{dN_1(t^*)}{dt}$  or  $\frac{dN_2(t^*)}{dt}$  is zero, since the only way a solution of (4.1) can leave region II is

by crossing  $l_1$  or  $l_2$ . Assume that  $\frac{dN_1(t^*)}{dt} = 0$ . Differentiation both sides of the first equation of (4.1) with respect to  $t$  and setting  $t = t^*$  gives

$$\frac{d^2 N_1(t^*)}{dt^2} = \frac{-a_1 \beta_1 N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt} \quad (4.12)$$

This quantity is positive. Hence  $N_1(t)$  has a minimum at  $t = t^*$ . However, this is impossible, since  $N_1(t)$  is increasing whenever a solution of  $N_1(t), N_2(t)$  of (4.1) is in region II.

Similarly, if  $\frac{dN_2(t^*)}{dt} = 0$ ,

$$\text{then } \frac{d^2 N_2(t^*)}{dt^2} = \frac{-(a_2 - q_2 E) \beta_2 N_2(t^*)}{k_2} \frac{dN_1(t^*)}{dt} \quad (4.13)$$

This quantity is negative, implying that  $N_2(t)$  has a maximum at  $t = t^*$ , but this is impossible, since  $N_2(t)$  is decreasing whenever a solution  $N_1(t), N_2(t)$  of (4.1) is in region II.

The previous argument shows that any solution  $N_1(t), N_2(t)$  of (4.1) which starts in region II at time  $t = t_0$  will remain in region II for all future time  $t \geq t_0$ . This implies that  $N_1(t)$  is monotonic increasing and  $N_2(t)$  is monotonic decreasing for  $t \geq t_0$ ; with  $N_1(t) < k_1$  and  $N_2(t) > k_2$ . Consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This in turn, implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$ ;  $(0, k_1)$  or  $(0, k_2)$ . Consequently,  $(\xi, \eta) = (\bar{N}_1, \bar{N}_2)$  and this proves Lemma 8.

**Lemma 9:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region III at time  $t = t_0$  will remain in this region for all future time  $t \geq t_0$ , and ultimately approach the equilibrium solution  $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$  (Fig.5).

**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) leaves region III at time  $t = t^*$ . Then either  $\frac{dN_1(t^*)}{dt}$  or  $\frac{dN_2(t^*)}{dt}$  is zero, since the only way a solution of (4.1) can leave region III is



by crossing  $l_1$  or  $l_2$ . Assume that  $\frac{dN_1(t^*)}{dt} = 0$ . Differentiation both sides of the first equation

of (4.1) with respect to  $t$  and setting  $t = t^*$  gives

$$\frac{d^2 N_1(t^*)}{dt^2} = \frac{-a_1 \beta_1 N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt} \quad (4.14)$$

This quantity is negative. Hence  $N_1(t)$  has a maximum at  $t = t^*$ . However, this is impossible, since  $N_1(t)$  is decreasing whenever a solution of  $N_1(t), N_2(t)$  of (4.1) is in region II.

Similarly, if  $\frac{dN_2(t^*)}{dt} = 0$ ,

$$\text{then } \frac{d^2 N_2(t^*)}{dt^2} = \frac{-(a_2 - q_2 E) \beta_2 N_2(t^*)}{k_2} \frac{dN_1(t^*)}{dt} \quad (4.15)$$

This quantity is positive, implying that  $N_2(t)$  has a minimum at  $t = t^*$ , but this is impossible, since  $N_2(t)$  is increasing whenever a solution  $N_1(t), N_2(t)$  of (4.1) is in region III.

The previous argument shows that any solution  $N_1(t), N_2(t)$  of (4.1) which starts in region III at time  $t = t_0$  will remain in region III for all future time  $t \geq t_0$ . This implies that  $N_1(t)$  is monotonic increasing and  $N_2(t)$  is monotonic decreasing for  $t \geq t_0$ ; with  $N_1(t) > k_1$  and  $N_2(t) < k_2$ . Consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This in turn, implies that  $(\xi, \eta)$  is an equilibrium point of (4.4.1). Now  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$ ;  $(0, k_1)$  or  $(0, k_2)$ . Consequently,  $(\xi, \eta) = (\bar{N}_1, \bar{N}_2)$  and this proves Lemma 9.

**Lemma 10:** Any solution of  $N_1(t), N_2(t)$  of (4.1) which starts in region VI at time  $t = t_0$  will remain in this region for all future time  $t \geq t_0$ , and ultimately approach the equilibrium solution  $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$  (Fig 5).

**Proof:** Suppose that a solution  $N_1(t), N_2(t)$  of (4.1) leaves region VI at time  $t = t^*$ . Then

either  $\frac{dN_1(t^*)}{dt}$  or  $\frac{dN_2(t^*)}{dt}$  is zero, since the only way a solution of (4.1) can leave region I is



by crossing  $l_1$  or  $l_2$ . Assume that  $\frac{dN_1(t^*)}{dt} = 0$ . Differentiation both sides of the first equation

of (4.4.1) with respect to  $t$  and setting  $t = t^*$  gives

$$\frac{d^2 N_1(t^*)}{dt} = \frac{-a_1 \beta_1 N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt} \quad (4.16)$$

This quantity is positive. Hence  $N_1(t)$  is monotonic decreasing and it has minimum whenever a solution of  $N_1(t), N_2(t)$  of (4.1) is in region VI.

Similarly, if  $\frac{dN_2(t^*)}{dt} = 0$ ,

$$\text{then } \frac{d^2 N_2(t^*)}{dt} = \frac{-(a_2 - q_2 E) \beta_2 N_2(t^*)}{k_2} \frac{dN_1(t^*)}{dt} \quad (4.17)$$

This quantity is positive, implying that  $N_2(t)$  is monotonic decreasing and it has minimum whenever a solution  $N_1(t), N_2(t)$  of (4.1) is in region VI.

If a solution  $N_1(t), N_2(t)$  of (4.1) remains in region VI for  $t \geq t_0$ , then both  $N_1(t)$  and  $N_2(t)$  are monotonic decreasing functions of time for  $t \geq t_0$ , with  $N_1(t) > k_1$  and  $N_2(t) > k_2$ , consequently, both  $N_1(t)$  and  $N_2(t)$  have limits  $\xi, \eta$  respectively, as  $t$  approaches infinity. This, in turn implies that  $(\xi, \eta)$  is an equilibrium point of (4.1). Now,  $(\xi, \eta)$  obviously cannot equal  $(0, 0)$ ;  $(k_1, 0)$  or  $(0, k_2)$ . Consequently  $(\xi, \eta) = (\bar{N}_1, \bar{N}_2)$ .

**Proof of Theorem:** Lemmas 7,8,9 and 10 state that every solution  $N_1(t), N_2(t)$  of (4.1) which starts in region I, II, III or VI at time  $t = t_0$  and remains there for all future time must also approach equilibrium solution  $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$  as  $t$  approaches infinity. Next, observe that any solution  $N_1(t), N_2(t)$  of (1) which starts on  $l_1$  or  $l_2$  must immediately afterwards enter regions I, II, III or VI. Finally the solution approaches the equilibrium solution  $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$ . This is illustrated in Fig. 6.



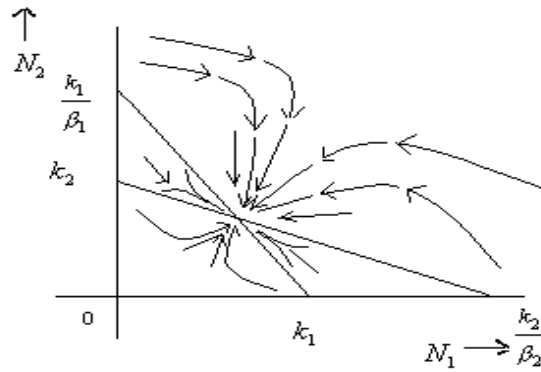


Fig.6

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