International Journal of Advanced Research in
Engineering and Applied Sciences ISSN: 2278-6252

SEQUENCE OF MAPS IN THE STRONGLY ITERATIVE WAY ON A COMPACT METRIC SPACE AND CHAOS OF THE UNIFORM LIMIT FUNCTION IN THE ITERATIVE WAY

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Abstract: *Recently, Tian and Chen have introduced a new concept of sequence of time invariant functions. In this paper we try to investigate the chaotic behavior of the uniform limit function in the iterative way of a sequence of continuous topologically transitive functions in the strongly iterative way in a compact interval. Surprisingly, we find that the uniform limit function in the iterative way is chaotic in the sense of Devaney. We also give some examples in the last section.*

Keywords: *Sequence of maps, Uniform convergence in the iterative way, Chaos in the sense of Devaney, Topological transitivity in the strongly iterative way.*

AMS Subject Classification 2010: *54H20, 37D45, 74H62.*

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1. INTRODUCTION

A dynamical system is a study of how physical and mathematical systems evolve with time, developed through the collective efforts of mathematicians and scientists in many different fields. A dynamical system includes the following components: a phase space *S* whose elements represent possible states of the system, time *t* (which may be discrete or continuous) and an evolution law (that is, a rule that allows determination of the state at time *t* from the knowledge of the states at all previous times). The applications and origins of dynamical systems lie in many different branches of science. Today the expression dynamical system is used as a synonym of nonlinear system of equations.

As indicated above dynamical systems may be divided into two broad categories, one is discrete dynamical system, where the time variable is discrete and the other is continuous dynamical system, where the time variable is continuous. During the last few decades there has been interesting development in the study of discrete systems of the form

$$x_{n+1} = f(x_n),\ n = 0,1,2,..........., \tag{1.1}$$

where (X,d) is a compact metric space and $f : X \to X$ is a continuous mapping. The basic target in the study of the system (1.1) is to understand the character of all the orbits $x, f(x), f(x^2),...........,$ for any $x \in X$.

Chaotic dynamical systems constitute a special class of dynamical systems, which received a great deal of attention from the past. A chaotic system is unpredictable and can not be broken down or decomposed into two invariant open subsets. Although there is no universally accepted mathematical definition of chaos, there are many wellknown definitions [3, 5, 6] in the world. But Devaney's [3] definition of chaos is one of the most popular one and it is also purely topological. In this paper we consider the definition of chaos in the sense of Devaney.

It is a reality that if a sequence of continuous functions converges uniformly, the uniform limit function is continuous. Also, the limit function of a uniformly convergent sequence of Riemann-integrable functions is itself Riemann-integrable. However, differentiability of the limit function is not assured by uniform convergence of a sequence of differentiable functions. In fact, uniform convergence of the derivatives is also needed. Therefore, it is meaningful and of interest to investigate the properties of the elements of an uniformly convergent sequence of functions that can be inherited by the limit functions. So it is an interesting line of research, to investigate which chaotic properties are preserved by uniform convergence. From the literature we have surveyed, it appears that the answer to this question is not yet thoroughly investigated. We have noticed that only some progress [2, 4] has been made in respect of uniform convergence of chaotic functions.

Tian and Chen [8] studied chaos of a sequence of time invariant continuous functions on a general metric space. The authors also introduced quite a few new concepts, such as 'chaos in the successive way in the sense of Devaney', 'chaos in the iterative way in the sense of Devaney' etc.

In this paper we have introduced the concepts of 'uniform convergence in the iterative way' and 'topological transitivity in the strongly iterative way'. We have proved that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions which is topologically transitive in the strongly iterative way in an infinite compact metric space (X, d), the uniform limit function in the iterative way is topologically transitive. Hence by the theorem of Vellekoop et al [9], we get that, if $f_n : X \to X, n \geq 1,$ is a sequence of continuous functions that converges uniformly to $f : X \to X$ in the iterative way, f is chaotic on X whenever the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive in the strongly iterative way, where *X* is any compact interval. We have provided an example to show that the denseness property of Devaney's definition is lost on the limit function. Also an example is given for illustrating the concept of uniform convergence in the iterative way.

2. MATHEMATICAL PRELIMINARIES

In this section we give some existing definition results and notations which are essential for the discussion in next sections. We also introduce some new definitions.

Throughout this paper we use the following mathematical notations.

i) The radius of any ball A is denoted by $Rad(A)$.

ii) If $\varepsilon > 0$ is arbitrary, we denote the ε-neighborhood of any point *x* by $S_{\varepsilon}(s)$.

iii) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from *X* to X, where (X, d) is a metric space. Then we denote $f_k \circ f_{k-1} \circ .... \circ f_1(x)$ by $F_k(x)$, for all $k \geq 1$ and $x \in X$.

iv) We denote the set of natural numbers by *N*.

v) Closure of any set *A* is denoted by $\overline{A}$.

vi) We also denote the complement of any set M by M^c.

In the following we state some definitions.

Definition 2.1 [8] (Orbit in the iterative way): Let (X,d) be a compact metric space and $x \in X$ be any point. Also let $f_n : X \to X$, $n \geq 1$, be a sequence of continuous functions. Then $\{x, f_1(x), f_2 \circ f_1(x), f_3 \circ f_2 \circ f_1(x), \ldots\ldots\ldots\ldots\ldots\}$ is called orbit of the sequence $\{f_n\}_{n=1}^{\infty}$ (starting at *x*) in the iterative way.

Definition 2.2 [7] (Uniform convergence): Let (X,d) be a metric space and $f_n : X \to X$, $n \geq 1$, be a sequence of continuous functions defined on X. Let $f : X \to X$ be a continuous function such that, $d(f_n(x), f(x)) < \varepsilon$, for all $n \geq n_0$ and for all $x \in X$, where n_0 is a positive integer (depending on ε only), then we say that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f. If $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f, we write $f_n \xrightarrow{uniformly} f$.

We now introduce the notion of uniform convergence in the iterative way.

Definition 2.3 (Uniform convergence in the iterative way): Let (X,d) be a metric space and $f_n : X \to X$, $n \geq 1$, be a sequence of continuous functions defined on X. Let $f : X \to X$ be a continuous function such that, $d(F_n(x), f(x)) < \varepsilon$, for all $n \geq n_0$ and for all $x \in X$, where n_0 is a positive integer (depending on ε only), then we say that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f in the iterative way. If $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f in the iterative way, we write $f_n \xrightarrow{uniformly} f$ in the iterative way.

It should be noted that if $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to *f* in the iterative way, the same $\{F_n\}_{n=1}^{\infty}$ is uniformly convergent to *f*.

An example of the above type of convergent sequence has been illustrated in the last section.

Definition 2.4 [3] (Sensitive dependence on initial conditions): A continuous map $f : X \to X$, where (X,d) is a metric space, has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $N(x)$ of x, there exist $y \in N(x)$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Definition 2.5 [3] (Topological transitivity): The function $f: X \to X$ is said to be topologically transitive if for any pair of non-empty open sets $K, L \subset X$ there exists $n > 0$ such that $f^n(K) \cap L \neq \phi$, where (X, d) is a metric space.

Definition 2.6 [3] (Dense set of periodic points): Let $f: X \to X$ be a continuous function on a metric space (X, d). If the set of all periodic points of f is dense in X, f is said to have a dense set of periodic points.

Definition 2.7 [3] (Chaos in the sense of Devaney): Let $f: X \to X$ be a continuous function on a metric space (X, d). Then *f* is said to be chaotic in the sense of Devaney if:

i) *f* is topologically transitive on *X*,

ii) the set of periodic points are dense in *X*,

iii) *f* has sensitive dependence on initial conditions.

Later it was shown by Banks et al [1], that the conditions (i) and (ii) together implies the condition (iii). So, property (iii) is redundant in the above definition.

Definition 2.8 [8] (Topological transitivity in the iterative way): Let (X, d) be a metric space and $f_n: X \to X$, $n \geq 1$, be a sequence of continuous functions. If, for any two non-empty open subsets U and V of X, there exists a positive integer k such that $F_k(U) \cap V \neq \phi$, the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be topologically transitive on X in the iterative way.

We give below a stronger version of the above definition.

Definition 2.9 (Topological transitivity in the strongly iterative way): Let (X, d) be a metric space and $f_n: X \to X$, $n \geq 1$, be a sequence of continuous functions. If,

i) for any two non-empty open subsets *U* and *V* of X, there exists a positive integer *k* such that $F_k(U) \cap V \neq \phi$ and

ii) for any two pair of distinct non-empty open subsets U_1, V_1 and U_2, V_2 of X there exist positive integers $k_1 \neq k_2$ such that, $F_{k_1}(U_1) \cap V_1 \neq \phi$ and $F_{k_2}(U_2) \cap V_2 \neq \phi$,

the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be topologically transitive on *X* in the strongly iterative way.

Definition 2.10 [8] (Sensitive dependence on initial conditions in the iterative way): Let (*X*, *d*) be a metric space and $f_n: X \to X$, $n \geq 1$, be a sequence of continuous functions. If there exists a constant $\delta > 0$ such that for any point $x \in X$ and any neighborhood $N(x)$ of *x*, there exist a point $y \in N(x)$ and a positive integer *k* such that $d(F_k(x), F_k(y)) > \delta$, the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to have sensitive dependence on initial conditions in the iterative way.

Definition 2.11 [7] (Well ordering property)**:** Every non-empty subset of the set of natural numbers has a least element.

We shall prove the following lemma which is required for proving the theorems.

Lemma **2.**1: Every infinite subset of the set of natural numbers is countable.

Proof: Let *N* be the set of natural numbers and A an infinite subset of *N*. Since *A* is a non-empty subset of *N*, it has a least element by well ordering property of *N*. Let a_1 be the least element of *A*. Then $A - \{a_1\}$ is non-empty; hence it has a least element by the same argument. Let a_2 be the least element of $A - \{a_1\}$. Similarly, $A - \{a_1, a_2\}$ is non-empty. So continuing this process we can write all elements of A as an infinite sequence, that is, $A = \{a_1, a_2, ...\}$. Hence *A* is countable. So we conclude that every infinite subset of the set of natural numbers is countable.

Lastly, we give the statement of an important result, which is known as Jacobi's Theorem.

Theorem 2.1 (Jacobi's Theorem [3]):

Let S^1 denotes the unit circle in the plane and $T_\lambda : S^1 \to S^1$ be defined by $T_\lambda(\theta) = \theta + 2\pi\lambda$, then each orbit T_λ is dense in S_1 if λ is irrational.

3. THE MAIN THEOREMS

Theorem 3.1: Let (X, d) be an infinite compact metric space and $\{f_n\}_{n=1}^{\infty}$ a continuous sequence of functions from X into *X* such that $f_n \xrightarrow{uniformly} f$ in the iterative way. If the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in the strongly iterative way, f is topologically transitive.

Proof: Let U_1 and V_1 be any two non-empty open subsets of X. Since the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in the strongly iterative way, there exists a positive integer

k_1 such that $F_{k_1}(U_1) \cap V_1 \neq \phi$, where the sequence $\{F_n\}_{n=1}^{\infty}$ is as defined earlier. Let $\varepsilon', \varepsilon'' > 0$. Since U_1 is a non-empty open subset of X, it has non-empty interior. For an interior point of U_1, we can take an open ball U of radius ε' centered at this interior point such that $U \subset U_1$. Next we take an open ball $U_2 \subset U$ such that $Rad(U_2) = \frac{\varepsilon'}{2}$ with center of U_2 same as that of U. We now consider the sets $\overline{U}_2$ and U^c. Then the distance between these two sets is at least $\frac{\varepsilon'}{2}$. Hence minimum distance between U_2 and U_1 is at least $\frac{\varepsilon'}{2}$. Similarly, we take an open ball $V_2 \subset V_1$ such that $Rad(V_2) = \frac{\varepsilon''}{2}$ and the minimum distance between V_2 and V_1 is at least $\frac{\varepsilon''}{2}$. Again, by topological transitivity in the strongly iterative way, there exists a positive integer k_2 different from k_1, such that $F_{k_2}(U_2) \cap V_2 \neq \phi$. We now take an open ball $U_3 \subset U_2$ such that $Rad(U_3) = \frac{\varepsilon'}{3}$ and the center of U_3 is same as that of U_2. Similarly, we take an open ball $V_3 \subset V_2$ such that $Rad(V_3) = \frac{\varepsilon''}{3}$ and the center of V_3 is same as that of V_2. Then by topological transitivity in the strongly iterative way, there exists a positive integer k_3 different from both k_1 and k_2, such that $F_{k_3}(U_3) \cap V_3 \neq \phi$. (This has been illustrated in Figure 1 and Figure 2.)

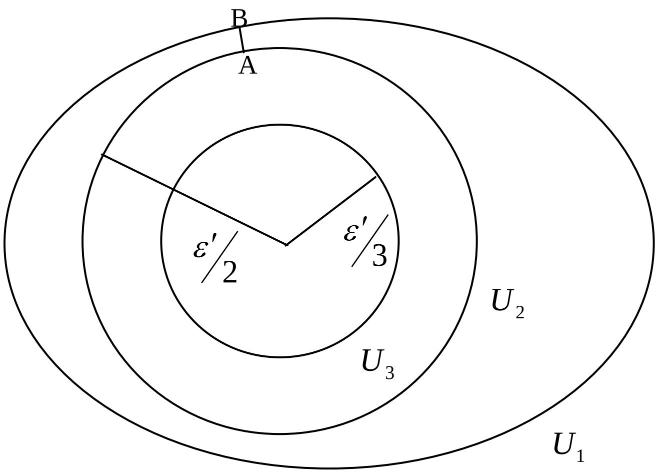

Figure 1

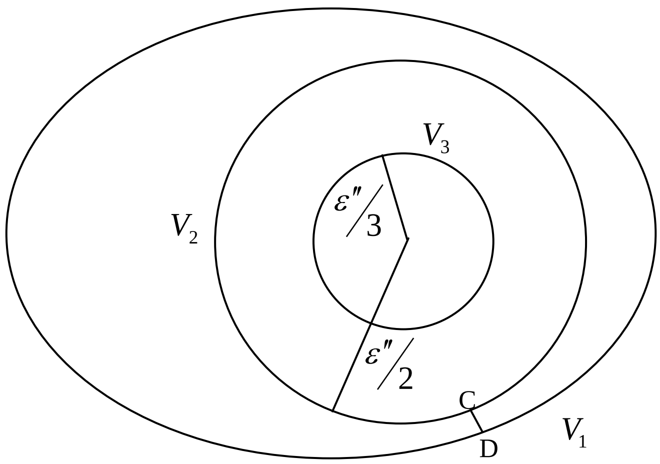

Figure 2

By our construction $AB \geq \frac{\varepsilon'}{2}$ and $CD \geq \frac{\varepsilon''}{2}$. Also, $F_{k_1}(U_1) \cap V_1 \neq \phi$, $F_{k_2}(U_2) \cap V_2 \neq \phi$, $F_{k_3}(U_3) \cap V_3 \neq \phi$ and so on.

Let us continue this process repeatedly. Then $\{k_n\}_{n=1}^{\infty}$ is an infinite subset of N and hence by Lemma 2.1, is countable. Therefore we can rearrange this set as a sequence by taking the lest element first, then the next lowest element and so on. We now denote this rearrangement by $\{k_{n'}\}_{n=1}^{\infty}$. Then $\{k_{n'}\}_{n=1}^{\infty}$ is a strictly monotonic increasing sequence of positive integers.

Here the following facts are noticeable:

a) U_i's and V_i's are non-empty open sets such that $U_{i+1} \subset U_i$ and $V_{i+1} \subset V_i$, for all $i \geq 1$.

b) There exists a sequence of positive integers $\{k_{n'}\}_{n=1}^{\infty}$ such that $F_{k_{n'}}(U_{n'}) \cap V_{n'} \neq \phi$, for all n'.

c) U_i's (and V_i's) are all open sets such that centres of U_i's (and V_i's) are same, for all $i \geq 2$.

d) By **a)** and **b)** it can be proved that $F_{k_{n'}}(U_1) \cap V_1 \neq \phi$, for all $k_{n'}$.

We now consider the sequence $\{F_{k_{n'}} : n' \geq 2\}$. Then $F_{k_{n'}} \xrightarrow{uniformly} f$, that is, $\{f_{k_{n'}} : n' \geq 2\}$ is uniformly convergent to f in the iterative way.

Then for $\varepsilon = \frac{\varepsilon''}{10}$, $d(F_{k_{n'}}(x), f(x)) < \varepsilon$, for all $n' \geq m'$ and for all $x \in X$, for some $m' \in N - \{1\}$. (3.1)

We now show that $f^l(U_1) \cap V_1 \neq \phi$, for an $l > 0$.

Let $y \in F_{k_{m'}}(U_{m'}) \cap V_{m'}$. (3.2)

Hence $y \in V_{m'}$ and $y \in F_{k_{m'}}(U_{m'})$. So there exists a $x \in U_{m'}$ such that $F_{k_{m'}}(x) = y$. Again from (3.1) we get $d(F_{k_{m'}}(x), f(x)) < \varepsilon$.

So $f(x) \in S_\varepsilon(F_{k_{m'}}(x))$, that is, $f(x) \in S_\varepsilon(y)$. (3.3)

From (3.2) we get $y \in F_{k_{m'}}(U_1)$ and $y \in V_2 \subset V_1$. Now $x \in U_{m'} \Rightarrow x \in U_1 \Rightarrow f(x) \in f(U_1)$.

Since $m' \neq 1$, by the definition of ε and (3.3) and also our construction above we get that $f(x) \in V_1$. Hence $f(U_1) \cap V_1 \neq \phi$.

So, we conclude that f is topologically transitive.

Hence by the application of the theorem of Vellekoop et al [9] and Theorem 3.1 as above we get our desired result.

Theorem 3.2 Let X be a compact interval of real numbers and $f_n : X \to X, n \geq 1,$ be a sequence of continuous functions that converges uniformly to $f : X \to X$ in the iterative way, then f is chaotic on X whenever $\{f_n\}_{n=1}^{\infty}$ is topologically transitive in the strongly iterative way.

4. CONCLUSIONS

Recently, Flores [4] has shown that, if $f_n : X \to X, n \geq 1,$ is a topologically transitive continuous sequence of functions on a compact metric space (X, d), the uniform limit function *f* is not necessarily topologically transitive and also he has given a sufficient condition for transitivity of *f* into a perfect metric space. But in this paper we have proved that, if $f_n : X \to X, n \geq 1,$ is a sequence of topologically transitive continuous functions in the strongly iterative way on an infinite compact metric space (X, d), the uniform limit function in the iterative way *f* is also topologically transitive (Theorem 3.1) in *X*. Hence by the theorem of Vellekoop et al, we can say that if $f_n : X \to X, n \geq 1,$ is a sequence of topologically transitive continuous functions in the strongly iterative way on a compact interval, the uniform limit function is chaotic. In [4], some additional conditions are assumed to prove that the limiting function is chaotic. Here no other conditions are assumed for proving f to be chaotic, in Theorem 3.2.

We now give an example to show that denseness of periodic points may be lost in the case of the limit function.

Example 4.1: We consider the sequence of translation maps $T_{\left(1+\frac{1}{2n}\right)^n} : S^1 \to S^1$ by

$T_{\left(1+\frac{1}{2n}\right)^n}(x) = x + 2\pi\left(1+\frac{1}{2n}\right)^n$, where S^1 is the unit circle on the plane. Now obviously $\left(1+\frac{1}{2n}\right)^n$ is a rational number, for all $n \geq 1$. Hence $\left(1+\frac{1}{2n}\right)^n = \frac{p}{q}$, where p, q are integers and $q \neq 0$. Then we get that $T^q_{\frac{p}{q}}(x) = x + 2\pi q.\frac{p}{q} = x + 2\pi p = x$. So all points of S^1 are periodic with periods *q*. Hence the set of all periodic points of S^1 is dense in S^1. But note that, $\underset{n\to\infty}{Lt}\left(1+\frac{1}{2n}\right)^n = \sqrt{e}$, where $\sqrt{e}$ is an irrational number.

Then by Jacobi's Theorem, that is, the Theorem 2.1, we get that $T_{\sqrt{e}}(x)$ has a dense orbit for each $x \in S^1$. Thus there is no periodic point. This proves that denseness of periodic points will be lost in the case of the limit function.

Lastly, we give an example of a uniformly convergent sequence in the iterative way.

Example 4.2 We consider the usual metric space. Let $f_n(x):[0,2] \to [0,2]$ be a sequence of continuous functions, defined by $f_n(x) = \left(1-\frac{1}{n}\right)x$, for $n \geq 2$ and $f_1(x) = x$. Then by simple calculation we get $F_n(x) = \frac{x}{n}$, for all $n \geq 1$. Hence $f_n(x):[0,2] \to [0,2]$ is a sequence of continuous functions such that $F_n \xrightarrow{uniformly} 0$. Consequently $f_n \xrightarrow{uniformly} 0$ in the iterative way.

AN OPEN PROBLEM

In this paper we have proved that the uniform limit function in the iterative way of a sequence of topologically transitive functions in the strongly iterative way is chaotic in a compact interval. In this context there arises an open problem. Is the uniform limit function of a sequence of topologically transitive functions in the strongly iterative way is chaotic in a compact interval? The answer to this question is yet to be investigated.

ACKNOWLEDGEMENTS

Indranil Bhaumik acknowledges his father Mr. Sadhan Chandra Bhaumik for his help in preparing the manuscript.

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